ELITE PROJECT - EXTENSION IN QUATERNION GROUP
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ABSTRACT
In this project, we have worked on Quaternions. We created the quaternion field and space. We have integrated quaternions with some examples of nature. Rotation of the earth is a quaternion ring, cyclooctane constructed by $\text{CH}_2$ and CH are similar to the quaternion octane by taking $\pm 1 = \pm i = \pm j = \pm k = \text{CH}_2$ and CH. The movement of a Cubesat is a group action on the quaternions and a cubesat itself is a quaternion space. Even the movement of a football is a rotation of the quaternions.

INTRODUCTION
Quatereion group consist of 8 elements namely $\pm 1$, $\pm i$, $\pm j$, $\pm k$, where $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$.

Q_8 is a non commutative group as $ij \neq ji$
PROPERTIES
1. $Q_8$ has 6 subgroups.
2. $Q_8$ is non-cyclic but every proper subgroup of $Q_8$ is cyclic.
3. Every proper subgroup of $Q_8$ is normal.
4. Every proper subgroup of $Q_8$ is abelian.
5. $Z(Q_8) = \{1,-1\}$.

QUATERNION RING
Definition:- A non-empty set $Q_8=\{a+bi+cj+dk \mid a,b,c,d \in R, i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, kj = -i, ik = -j\}$, where $R$ is any ring, is the ring as
The set $Q_8$
1. is an abelian group under addition.
2. is a semi group under multiplication.
3. satisfies left and right distributive properties.
And this ring is known as quaternion ring.

QUATERNION FIELD
A non-empty set $Q_8=\{a+bi+cj+dk \mid a,b,c,d \in F, i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, kj = -i, ik = -j\}$, where $F$ is a field, is a skew field as
the set $Q_8$
1. is an abelian group under addition.
2. is a semi group under multiplication.
3. satisfies left and right distributive properties.
We have the multiplicative identity of $(Q_8, \cdot)$ is $1 = 1 + 0 (i+j+k)$
with the inverse of $a = 1/a \forall a \neq 0$.
and inverse of $a + bi = (a - bi)/(a^2+b^2)$ where either $a$ or $b$ is nonzero.
and inverse of $a + bi + cj + dk = (a-bi-cj-dk)/(a^2+b^2+c^2+d^2)$ where either $a,b,c$ or $d$ is nonzero.

Since every element of $Q_8$ has its inverse in $Q_8$, $Q_8$ is a division ring as $Q_8$ is non-abelian. Therefore,
$Q_8$ is a skew field. This is to be called Quaternion field.

QUATERNION SPACE
Vector Space:- Let $S$ be a non-empty set, then the set $S$ with two binary operations vector addition and scalar multiplication ($\cdot$), is said to be a vector space over a field $F(=R)$ if the following properties hold for every $A,B,C \in S$ and $a,b \in R :-$
1. $(S,+) \text{ forms an abelian group.}$
2. $S$ is closed under scalar multiplication.
3. $S$ satisfies the associative property with scalar multiplication.
4. For $\forall x \in S, 1.x=x$.
6. $S$ satisfies the distributive property of scalar multiplication over vector addition.
7. $S$ satisfies the distributive property of vector multiplication over scalar addition.

$Q_8=\{a+bi+cj+dk \mid a,b,c,d \in F, i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, kj = -i, ik = -j\}$, where $F$ is a field}, satisfies the above mentioned properties, therefore it forms a vector space over $F$. This is to be called quaternion space.
APPLICATIONS

APPROACH IN CHEMISTRY

Conformation of organic cyclic compounds: - In organic compounds such as cyclooctatetraene (C₈ H₈) and cyclooctane (CH₂)₈. We can relate with a closed figure with 8 vertices where every Carbon can be compared with a distinct element of Q₈.

\[
\begin{align*}
 & \text{Cyclooctane} \\
 & \text{Cyclooctatetraene}
\end{align*}
\]

Both figures can be compared with this

\[
\begin{align*}
1 & \quad i \\
-j & \quad j \\
-k & \quad k \\
-1 & \quad -1
\end{align*}
\]

Where \(|±1| = |±i| = |±j| = |±k| = \text{one mole of } CH₂ \text{ in first case and one mole of } CH \text{ in second case, and } ±1 = ±i = ±j = ±k = CH₂ \text{ in first case and } CH \text{ in second case}

APPROACH IN GEOGRAPHY

Globe: - A globe is a 3-D spherical, scale model of earth or other celestial body such as a planet or moon. We know that globe is a model of earth, where line drawn by joining north pole and south pole plays the role of axis of rotation.

\[
\text{We can arrange the quaternions in the form given below}
\]
This form is similar to the globe, though globe is bent with an angle of $231/2^\circ$, where $\pm 1$ stands for two poles and $\pm i, \pm j, \pm k$ are the points at equator line. If we rotate earth in step of $72^\circ$, then every point will at the place as it was initially, after 5th rotation as $x^5=x, \forall x \in Q_8$

From the top of the axis, rotations will look like figures given below:-

$R_0$ $\rightarrow$ $R_{72}$ $\rightarrow$ $R_{144}$ $\rightarrow$ $R_{216}$ $\rightarrow$ $R_{288}$
GROUP ACTION

Definition: A group action of a group $G$ on a set $A$ is a map from $G \times A$ to $A$ satisfying the following properties:

1. $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G, a \in A$, and
2. $1 \cdot a = a$, for all $a \in A$

Let the group $G$ acts on a set $A$. For each fixed $g \in G$, map $\sigma_g$ defined by

$\sigma_g : A \to A$

$\sigma_g(a) = g \cdot a$

$\sigma 1(a) = 1 \cdot a = a$, where $a \in Q_8$

$(1 - 1)(i - i)(j - j)(k - k)$

$\sigma -1(a) = -1 \cdot a$, where $a \in Q_8$

$(1 i - 1 - i)(j k - j - k)$

$\sigma i(a) = i \cdot a$, where $a \in Q_8$
$\circ -i(a) = -i \cdot a$, where $a \in Q_8$

$\circ j(a) = j \cdot a$, where $a \in Q_8$

$\circ -j(a) = -j \cdot a$, where $a \in Q_8$

$\circ k(a) = k \cdot a$, where $a \in Q_8$
\( \sigma_k(a) = -k \cdot a \), where \( a \in Q_8 \)

**Approach in Physics**

**Cubesat** :- A cubesat is a type of small satellite for space research that is made up of multiple 10×10×10 cm cubic units. Cubesats have a mass no more than 1.33 Kg per unit. Cubesats are most commonly put in orbit by deployers on the International Space Station. Basically, it has 8 vertices that’s why it can be compared to \( Q_8 \).

The movement of the above satellite can be studied by the group action of \( Q_8 \) on itself. This can be understood by labelling 8 vertices of cubesat as \( \pm 1, \pm i, \pm j, \pm k \) same as the above diagram of \( \sigma_i \) and \( \sigma_i^- \). Cubesat will rotate in clockwise direction and anticlockwise through group action \( \sigma_i \) and \( \sigma_i^- \).
**Application of rotation of Qs in sports:** The movement of football can be compared with group action of $Q_8$ on itself. This can be understood by the rotations due to group actions $\sigma_k$ and $\sigma_{-k}$. 

\[
\begin{align*}
\sigma_k(x) &\quad \text{for } k = 1, 2, 3, 4, 6, 7, 15, 16, \\
\sigma_{-k}(x) &\quad \text{for } k = 1, 2, 3, 4, 6, 7, 15, 16.
\end{align*}
\]

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**CONCLUSION**
We have extended Quaternion ring to quaternion field and to quaternion space. Apart from this, we went through some of its applications, which behaved as some of the applications of Physics, Chemistry, and Geography.

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