Poisson Distribution

Introduction

- We consider another example of discrete probability distributions called The Poisson distribution.

- The Poisson distribution is widely used in the study of business and industrial processes.

- This distribution was discovered by the French mathematician Simon Denis Poisson.

When n is large, the calculation of binomial probabilities with the formula that was derived in the last section, will usually involve a prohibitive amount of work.

For example, if in the binomial process

$$n = 7000,\ x = 25 \text{ and p} = 0.001$$

then we need to calculate

$$b(x; n, p) = {}^{n}C_{x}\, p^{x} (1 - p)^{n-x}$$

$$= {}^{7000}C_{25}\, (0.001)^{25} (0.999)^{6975}$$

- The Poisson distribution comes into play in situations in which discrete events are being observed in some continuous interval of time or space.

- The random variable of interest is the number of occurrences of the observation period.

- The given time interval may be of any length, such as a minute, a day, a week, a month or even a year.
- The Poisson experiment might generate observations for the random variable *X*, representing the number of telephone calls per hour received by an office,

- the number of weeks the Universities are closed down due to strike actions,
- or the number of postponed games due to rain during a football season.

A Poisson experiment is one that possesses the following properties:

1. The number of successes occurring in one time interval of specified region are independent of those occurring in any other disjoint time interval or region of space.

- The probability of a single success occurring during a very short time interval is proportional to the length of the time interval and does not depend on the number of successes occurring between this time interval

- The probability of more than one success occurring in such a short interval is negligible.

- When we observe the occurrence of discrete events in a continuous interval, we are observing what is called a Poisson process.
- Each Poisson process is characterised by one parameter, which we denote by λ .

- This parameter gives the average number of occurrences of the event in a unit interval

- The probability distribution of the Poisson variable *X* is called the Poisson distribution and will be denoted by $p(x;\lambda)$,

Definition The probability distribution of the Poisson random variable *X*, representing the number of success occurring in a given time interval or specified region is given by

$$p(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x=0, 1, 2, \ldots,.$$

where λ is the average number of successes occurring in the given time interval or specified region and

$$e = 2.71828.$$

Statistical tables contain Poisson probability sums:

$$p(x;\lambda) = \sum_{x=0}^{n} p(x;\lambda)$$

Example.3.4

The average number of taxi drivers wearing seatbelts at Winneba junction in one hour is four. What is the probability that six drivers will be found to wear seatbelts in a given hour.

Example3.5

Suppose the average number of oil tankers arriving each day at Tema harbour is 10 and the facilities at the harbour can handle at most 15 tankers per day. What is the probability that on a given day tankers will have to be sent away?

Theorem The mean and the variance of the Poisson distribution both have the value λ .

Theorem Let X be a binomial random variable with probability distribution $b(x;n,p)$ When $n \to \infty$,

$p \to 0$, and $\lambda = np$, remains constant,

,

$$b(x;n,p) \to p(x;\lambda)$$

This theorem states that if *n* is large and p is small and remains constant, then the binomial distribution can be approximated by the Poison distribution.

An acceptable rule of thumb is to use this approximation if $n \geq 20$ and $p \leq 0.05$ If $n \geq 100$ the approximation is generally excellent, so long as $np \leq 10$.

Example 3.6

A publishing Company of mathematics books Ghana Mathematics Group (GMG), takes pains to ensure that its books are free of typographical errors. The probability of any given page containing at least one such error is 0.005 and errors are independent from page to page. What is the probability that one of its 400 page book will contain

- exactly one page with errors?
- at most three pages with errors?

Example 3.7

It is known that 5% of the books bound at a certain bindery have defective bindings. Find the probability that 2 of 100 books bound by this bindery will have defective bindings using

1. the formula for the binomial distribution
2. the Poisson approximation to the binomial distribution.