# Power Series Solutions about a Regular Singular Point

# Method of Frobenius

Let $x = x_0$ be a regular singular point of the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

where $a_0(x),\ a_1(x),\ a_2(x)$

are polynomials

Then (1) has Frobenius-type solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\lambda} \qquad (2)$$

where the series converges for all $x$ such that $|x - x_0| < R$ and R is the distance from $x_0$ to the nearest singularity

The expression given in (2) is called a *Frobenius-type expansion* of $y(x)$ about $x_0$

The first and second derivatives of $y(x)$ can be computed term-wise

$$y'(x) = \sum_{n=0}^{\infty} (n+\lambda) a_n (x - x_0)^{n+\lambda-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1)(x - x_0)^{n+\lambda-2}$$

The following steps are useful to take when using the Frobenius method

1. expand the polynomial $a_0$, $a_1$, and $a_2$ in terms of $(x - x_0)$
2. substitute the Frobenuis-type expansions for $y, y'$, and $y''$ into equation (1)

- multiplying the polynomials with the series
- change the summation index n so that the series can be added term by term
- combine the resulting series into one series

- now choose $\lambda$ so that the coefficient of the lowest power of $(x - x_0)$ is zero. The equation obtained in this process is called the **indicial equation**, and the roots $\lambda$ of this equation are called the **indicial roots**.

- Once the indicial roots are computed, they can be substituted for $\lambda$, and a recurrence formula for the $a_n$'s can be obtained by setting the coefficients of the remaining powers of $(x - x_0)$ equal to zero.
- This procedure is called the *method of Frobenius*.

Let one of the two linearly independent solutions of (2) be in the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\lambda_1}$$

and let $\lambda_1$ and $\lambda_2$ be the roots of the indicial equation of (1) at $x_0$ such that $\lambda_1 \geq \lambda_2$ and both roots are real numbers

Then the second linearly independent solution $y_2(x)$ can be considered in the following general forms:

**Case 1**: If $\lambda_1 - \lambda_2 \neq$ integer

$$y_2(x) = \left|x - x_0\right|^{\lambda_2} \sum_{n=0}^{\infty} c_n \left|x - x_0\right|^n$$

**Case 2**: If $\lambda_1 = \lambda_2$

Then $y_2(x) = y_1(x)\ln|x - x_0| + \sum_{n=0}^{\infty} c_n (x - x_0)^{n+\lambda_2}$

**Case 3**: If $\lambda_1 - \lambda_2 =$ positive integer, then

$$y_2(x) = cy_1(x)\ln|x - x_0| + \sum_{n=0}^{\infty} c_n (x - x_0)^{n+\lambda_2}$$

Example

Find the general solution of the equation

$$2xy''+y'-2y=0$$

near the point $x=0$