Homogeneous Differential Equations

Definition A function $f(x,y)$ is said to be homogeneous of degree *n* in the variables *x* and *y* if for any λ, the following identity is true

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

Example 3.1

Determine if the function $f(x,y) = xy + y^2$ is homogenous.

Example 3.2

Given that $f(x,y) = \sqrt[3]{x^3 + y^3}$ determine the homogeneity of this function.

Example 3.3

A function is defined by $f(x, y) = x^3 + xy^2 e^{\frac{x}{y}}$

Is this a homogeneous function?

Example 3.4

A function is defined by $f(x, y) = x + xy$

Determine the homogeneity and the degree of this function.

Example 3.5

Given that the function $f(x,y) = \dfrac{x^2 - y^2}{xy}$

Determine the homogeneity and the degree of this function.

Definition: A differential equation of the first order $\frac{dy}{dx} = f(x, y)$ is called homogeneous in *x* and *y* if the function $f(x, y)$ is a homogenous function of degree zero in *x* and *y*.

From the examples above determine the functions that can form homogeneous equations.

Definition: A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$

is homogeneous if both $M(x, y)$ and $N(x, y)$ are homogenous functions of the same degree.

Solutions to Homogeneous Equations

Theorem: Consider the first order homogenous differential equation $\frac{dy}{dx} = f(x, y)$.
Then the change of variables $y = ux$ transforms it into a separable equation in the variables *u* and *x*.

Proof:

The given equation is said to be homogeneous ad therefore by definition we have

$$f(x, y) = f(\lambda x, \lambda y) = \lambda^0 f(x, y)$$

Let $\lambda = \dfrac{1}{x}$ then $f(x, y) = f\left(1, \dfrac{y}{x}\right)$

This implies that $\frac{dy}{dx} = f\left(1,\ \frac{y}{x}\right)$

Making the substitution $y = ux$ we find that

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

Equating them, we get $u + x\frac{du}{dx} = f(1,\ u)$

We simplify to get $$\frac{dx}{x} = \frac{du}{f(1,\ u) - u}$$

Then $$\int \frac{dx}{x} = \int \frac{du}{f(1,\ u) - u} + \ln c$$

We get $$x = ce^{\int \frac{du}{f(1,\ u) - u}}$$

Example 3.6

Consider the equation $y' = \dfrac{xy}{x^2 - y^2}$

Determine whether the function on the right is a zero-degree homogenous function. If it is, determine its general solution.

Example 3.7

Solve the differential equation $y' = \dfrac{y+x}{x}$

Example 3.8

Find the general solution of the equation

$$(x^4 + y^4)dx + 2x^3 y dy = 0$$

Equations Reducible to Homogeneous

Consider the equation

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1 x + b_1 y + c_1}$$

We see that equations of this form are not homogenous.