A NEW SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

J. K. PRAJAPAT, AMBUJ KUMAR MISHRA, AND P. K. TONDAN

Abstract. In this work, we introduce and investigate an interesting subclass $\mathcal{X}_t(\gamma)$ of analytic and close-to-convex functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{X}_t(\gamma)$, we drive several properties including (for example) the coefficient estimates, distortion theorems, covering theorems and radius of convexity.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}, \mathcal{S}^*$ and $\mathcal{K}$ be the usual classes of function which are also univalent, starlike and convex, respectively. We also denote by $\mathcal{S}^*(\gamma)$ the class of starlike function of order γ, where $0 \leq \gamma < 1$.

Definition 1.1. *If f and g are two analytic functions in $\mathbb{U}$, then f is said to be* subordinate *to g, and write $f(z) \prec g(z)$, if there exists a function w analytic in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in* U*, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ in $\mathbb{U}$.*

Cho and Zhou [2] introduced following subclass $\mathcal{K}_s$ of analytic function, which indeed a subclass of close-to-convex functions.

Definition 1.2. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_s$, if there exist a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that*

$$\Re\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > 0, \quad z \in \mathbb{U}. \tag{1.2}$$

Recently, Knwalczyk and Leś-Bomba [3] extended Definition 1.2, by introducing the following subclass of analytic functions.

Definition 1.3. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_s(\gamma)$, $0 \leq \gamma < 1$, if there exist a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that*

$$\Re\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > \gamma, \quad z \in \mathbb{U}. \tag{1.3}$$

Motivated by above defined function classes, we introduce the following subclass of analytic functions.

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic Functions, Univalent Function, Starlike Functions, Close-to-Convex Function.

Definition 1.4. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{X}_t(\gamma)$ ($|t| \leq 1, t \neq 0, 0 \leq \gamma < 1$), if there exist a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that*

$$\Re\left(\frac{tz^2 f'(z)}{g(z)g(tz)}\right) > \gamma, \quad z \in \mathbb{U}. \tag{1.4}$$

In terms of subordination (1.4) can be written as

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1+(1-2\gamma)z}{1-z}, \quad z \in \mathbb{U}. \tag{1.5}$$

By simple calculation we see that inequality (1.5) is equivalent to

$$\left|\frac{tz^2 f'(z)}{g(z)g(tz)} - 1\right| < \left|\frac{tz^2 f'(z)}{g(z)g(tz)} + 1 - 2\gamma\right| \tag{1.6}$$

We see that

$$\mathcal{X}_{-1}(\gamma) = \mathcal{K}_s(\gamma) \quad \text{and} \quad \mathcal{X}_{-1}(0) = \mathcal{K}_s.$$

We now present an example of functions belonging to this class.

Example 1.1. The function

$$f_1(z) = \frac{2\gamma - 1 - t}{(t-1)^2} \ln \frac{1-tz}{1-z} - \frac{2(1-2\gamma)z}{(1-t)(1-z)}, \quad z \in \mathbb{U}. \tag{1.7}$$

belongs to the class $\mathcal{X}_t(\gamma)$. Indeed, f_1 is analytic in $\mathbb{U}$ and $f_1(0) = 0$. Moreover,

$$f_1'(z) = \frac{1+(1-2\gamma)z}{(1-tz)(1-z)^2}, \quad z \in \mathbb{U}.$$

If we put

$$g_1(z) = \frac{z}{1-z}, \quad z \in \mathbb{U}, \tag{1.8}$$

then $g_1 \in \mathcal{S}^*(\frac{1}{2})$ and

$$\Re\left(\frac{tz^2 f'(z)}{g(z)g(tz)}\right) = \Re\left(\frac{1+(1-2\gamma)z}{1-z}\right) > \gamma, \quad z \in \mathbb{U}.$$

This means that $f_1 \in \mathcal{X}_t(\gamma)$ and is generated by g_1.

Cho and Zhou [2] and Knwalczyk and Leś-Bomba [3], have obtained properties for the function classes $\mathcal{K}_s$ and $\mathcal{K}_s(\gamma)$, respectively. Moreover, some other interesting subclasses of $\mathcal{A}$ related to the function classes $\mathcal{K}_s$ and $\mathcal{K}_s(\gamma)$ were considered in [4, 5]. In the present paper, we obtained coefficient estimates, distortion theorems, covering theorems and radius of convexity of the function class defined by (1.4).

2. Coefficient Inequalities

We first prove the following result.

Theorem 2.1. *Let $g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ and given by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{U}, \tag{2.1}$$

If we put

$$F(z)=\frac{g(z)\,g(tz)}{tz}=z+\sum_{n=2}^{\infty}c_nz^n,\quad z\in\mathbb{U}, \tag{2.2}$$

then

$$c_n=b_n+b_2b_{n-1}t+b_3b_{n-2}t^2+...+b_{n-1}b_2t^{n-2}+b_nt^{n-1}, \tag{2.3}$$

and $F(z)\in\mathcal{S}^*$.

Proof. Result (2.2) can be found easily. Also $|tz|\leq|z|<1$, then from the definitions of starlike function, we have

$$\Re\left(\frac{zg'(z)}{g(z)}\right)>\frac{1}{2}\quad\text{and}\quad\Re\left(\frac{tz\;g'(tz)}{g(tz)}\right)>\frac{1}{2}.$$

Therefore

$$\Re\left(\frac{zF'(z)}{F(z)}\right)=\Re\left(\frac{zg'(z)}{g(z)}\right)+\Re\left(\frac{tz\;g'(tz)}{g(tz)}\right)-1$$
$$>\frac{1}{2}+\frac{1}{2}-1=0.$$

This proves the Theorem 2.1. □

Remark 2.1. From the definition of the class $\mathcal{X}_t(\gamma)$ and Theorem 2.1, we have

$$\Re\left(\frac{zf'(z)}{F(z)}\right)>\gamma\qquad(0\leq\gamma<1;\;z\in\mathbb{U}),$$

thus

$$\mathcal{X}_t(\gamma)\;\subset\;\mathcal{K}_s(\gamma)\;\subset\;\mathcal{K}_s\;\subset\;\mathcal{S}.$$

Theorem 2.2. *Let* $g(z)\in\mathcal{S}^*\left(\frac{1}{2}\right)$ *be a function given by (2.1)and* $0\leq\gamma<1$
If an analytic function f in U defined by (1.1) satisfies the inequality

$$\sum_{n=2}^{\infty}2n\,|a_n|+(|1-2\gamma|+1)\sum_{n=2}^{\infty}|c_n|\leq2(1-\gamma) \tag{2.4}$$

where for n = 2,3,4 ......, the coefficient of c_n *are given by (2.3), then* $f\in\mathcal{X}_t(\gamma)$ *and it is generated by g. In particular if*

$$\sum_{n=2}^{\infty}n\,|a_n|\leq1-\gamma$$

Proof. We set for f given by (1.1)and g defined by (2.1)

$$A=\left|zf'(z)-\frac{g(z)g(tz)}{tz}\right|-\left|zf'(z)-(1-2\gamma)\frac{g(z)g(tz)}{tz}\right|$$

$$=\left|\sum_{n=2}^{n=\infty}na_nz^n-\sum_{n=2}^{n=\infty}c_nz^n\right|-\left|(2-\gamma)z+\sum_{n=2}^{\infty}na_nz^n+(1-2\gamma)\sum_{n=2}^{n=\infty}c_nz^n\right| \tag{2.5}$$

hence for z ∈ $\mathbb{U}$,we have the inequality

$$A \leq \sum_{n=2}^{n=\infty} n|a_n||z|^n + \sum_{n=2}^{n=\infty} |c_n||z|^n - \left((2-2\gamma)\,|z|^n - \sum_{n=2}^{n=\infty} n|a_n||z|^n - |1-2\gamma| \sum_{n=2}^{n=\infty} |c_n||z|^n \right)$$
$$= -(2-2\gamma)|z| + \sum_{n=2}^{n=\infty} 2n|a_n||z|^n + (|1-2\gamma|+1) \sum_{n=2}^{n=\infty} |c_n||z|^n$$
$$< \left(-(2-2\gamma) + \sum_{n=2}^{n=\infty} 2n|a_n| + (|1-2\gamma|+1) \sum_{n=2}^{n=\infty} |c_n| \right) |z|$$
$$\leq 0$$

From the above calculation we obtain that $A < 0$. Thus by (2.5) we have

$$\left| zf'(z) - \frac{g(z)g(tz)}{tz} \right| < \left| zf'(z) + \frac{-(1-2\gamma)g(z)g(tz)}{tz} \right| \qquad z \in \mathbb{U}$$

which is equivalent to inequality (1.6)and also to the inequality (1.4). Thus$f \in \mathcal{X}_t(\gamma)$and it complete the proof.

Theorem 2.3. *Let $0 \leq \gamma < 1$. Suppose that an analytic function f given by (1.1) and $g \in S^*(\frac{1}{2})$ given by (2.1) are such that condition (1.4) holds . Then for n = 2,3,..........we have*

$$n^2|a_n|^2 - 4|1-\gamma|^2 \leq \left(|2\gamma-1|^2 - 1\right) \sum_{k=2}^{k=n} |c_k|^2 \tag{2.6}$$

where c_n is defined by (2.3). In particular , if g(z)= z, then

$$n|a_n| \leq 2(1-\gamma)$$

Proof. Since$f \in \mathcal{X}_t(\gamma)$, for some $g \in S^*(\frac{1}{2})$ the inequality (1.6) holds. From the lemma, which was proved by Owa(sec[6]) with $\alpha = \beta = 1$, we have

$$\frac{zf'(z)}{g(z)} = \frac{1+(2\gamma-1)z\phi(z)}{1+z\phi(z)} \qquad z \in \mathbb{U}$$

where ϕ is an analytic function in $U, |\phi(z)| \leq 1$, for $z \in \mathbb{U}$ and g is given by (2.1). Then

$$\left(zf'(z) - (2\gamma-1)F(z)\right) z\phi(z) = F(z) - zf'(z)$$

Now if we put $z\phi(z) = \sum_{n=1}^{n=\infty} v_n z^n$

we see that $|z\phi(z)| \leq |z|$, for $z \in \mathbb{U}$. Thus

$$\left((2-2\gamma)z + \sum_{n=2}^{n=\infty} na_n z^n - (2\gamma-1) \sum_{n=2}^{n=\infty} c_n z^n \right) \sum_{n=1}^{n=\infty} v_n z^n = \sum_{n=2}^{n=\infty} a_n z^n - \sum_{n=2}^{n=\infty} na_n z^n \tag{2.7}$$

we compare co-efficients in (2.7). Hence we can write for n ≥ 2

$$\left((2-2\gamma)z + \sum_{k=2}^{k=n-1} ka_k z^k - (2\gamma-1) \sum_{k=2}^{k=n} c_k z^k \right) z\phi(z) = \sum_{k=2}^{k=n} c_k z^k - \sum_{k=2}^{k=n} ka_k z^k + \sum_{k=n+1}^{k=\infty} d_k z^k$$

Then we square the modulus of both sides of the above inequality and the we integrate along $|z| = r < 1$. After using the fact that $|z\phi(z)| \leq |z| < 1$, we obtain

$$\sum_{k=2}^{k=n}|c_k|^2r^{2k}+\sum_{k=2}^{k=n}|ka_k|^2r^{2k}+\sum_{k=n+1}^{k=\infty}|d_k|^2r^{2k}<|2-2\gamma|^2r^2+\sum_{k=2}^{k=n-1}|ka_k|^2r^{2k}+|2\gamma-1|^2\sum_{k=2}^{k=n}|c_k|^2r^{2k}$$

Letting r $\to 1, wehave$

$$\sum_{k=2}^{k=n}|c_k|^2+\sum_{k=2}^{k=n}|ka_k|^2\leq|2-2\gamma|^2+\sum_{k=2}^{k=n-1}|ka_k|^2+|2\gamma-1|^2\sum_{k=2}^{k=n}|c_k|^2$$

Hence

$$k^2|a_k|^2-4(1-\gamma)^2\leq\left(|2\gamma-1|^2-1|\right)\sum_{k=2}^{k=n}|c_k|^2$$

Thus we have the inequality (2.6), which finishes the proof. □

Theorem 2.4. *Let $0\leq\gamma<1$. If the function $f\in\mathcal{X}_t(\gamma)$, then*

$$|a_n|\ \leq\ \frac{1}{n}\left\{|c_n|+2(1-\gamma)\left(1+\sum_{k=2}^{n-1}|c_k|\right)\right\},\quad k\in\mathbb{N}. \tag{2.8}$$

Proof. By setting

$$\frac{1}{1-\gamma}\left(\frac{zf'(z)}{F(z)}-\gamma\right)=h(z),\quad z\in\mathbb{U}, \tag{2.9}$$

or equivalently

$$zf'(z)=\left[1+(1-\gamma)(h(z)-1)\right]F(z), \tag{2.10}$$

we get

$$h(z)=1+d_1z+d_2z^2+\cdots,\quad z\in\mathbb{U}, \tag{2.11}$$

where $\Re(h(z))>0$. Now using (2.2) and (2.10) in (2.11), we get

$$\begin{aligned}
2a_2&=(1-\gamma)d_1+c_2\\
3a_3&=(1-\gamma)(d_2+d_1c_2)+c_3\\
4a_4&=(1-\gamma)(d_3+d_2c_2+d_1c_3)+c_4\\
&\vdots\\
na_n&=(1-\gamma)(d_{n-1}+d_{n-2}c_2+\cdots+d_1c_{n-1})+c_n.
\end{aligned}$$

Since $\Re(h(z))>0$, then $|d_n|\leq 2,\ n\in\mathbb{N}$. Using this property, we get

$$\begin{aligned}
2|a_2|&\leq|c_2|+2(1-\gamma),\\
3|a_3|&\leq|c_3|+2(1-\gamma)\left\{1+|c_2|\right\}
\end{aligned}$$

and

$$4|a_4|\leq|c_4|+2(1-\gamma)\left\{1+|c_2|+|c_3|\right\},$$

respectively. Using the principle of mathematical induction, we obtain (2.4). This completes proof of Theorem 2.4 □

Corollary 2.1. *Let $0\leq\gamma<1$. If the function $f\in\mathcal{X}_t(\gamma)$, then*

$$|a_n|\ \leq\ 1+(n-1)(1-\gamma). \tag{2.12}$$

Proof. From Theorem 1, we know that $F(z) \in \mathcal{S}^*$, thus $|c_n| \leq n$. The assertion (2.12), can now easily derived from Theorem 2. $\square$

Remark 2.2. Setting $t = -1$ in (3

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), we find that

$$\begin{aligned} &c_{2n} = 0,\ n \in \mathbb{N}, \\ &c_3 = 2b_3 - b_2^2,\ \ c_5 = 2b_5 - 2b_2b_4 + b_3^2,\ \ c_7 = 2b_7 - 2b_2b_6 + 2b_3b_5 - b_4^2, \cdots \end{aligned}$$

thus

$$c_{2n-1} = B_{2n-1},\ n = 2, 3, \cdots,$$

where

$$B_{2n-1} = 2b_{2n-1} - 2b_2b_{2n-2} + \cdots + (-1)^n 2b_{n-1}b_{n+1} + (-1)^{n+1}b_n^2,\ n = 2, 3, \cdots.$$

Therefore, setting $t = -1$ in Theorem 2.2 and using the known inequality [2, Theorem B]

$$|B_{2n-1}| \leq 1,\ n = 2, 3, \cdots,$$

we get the corresponding result due to Geo and Zhou [2].

Theorem 2.5. *Let* $0 \leq \gamma < 1$. *If the function* $f \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty} \{|na_n - c_n| + (1-\gamma)|c_n|\} \ \leq\ 1 - \gamma,\ \ z \in \mathbb{U}, \tag{2.13}$$

then $f(z) \in \mathcal{X}_t(\gamma)$

Proof. If f satisfies (1.2), then

$$\left| \frac{tz^2 f'(z)}{g(z)\ g(tz)} - 1 \right| < 1 - \gamma,\ \ z \in \mathbb{U}. \tag{2.14}$$

Evidently, since

$$\begin{aligned} \frac{tz^2 f'(z)}{g(z)\ g(tz)} - 1 &= \frac{z + \sum_{n=2}^{\infty} n\ a_n z^n}{z + \sum_{n=2}^{\infty} c_n z^n} - 1 \\ &= \frac{\sum_{n=2}^{\infty} (na_n - c_n) z^{n-1}}{1 + \sum_{n=2}^{\infty} c_n z^{n-1}}, \end{aligned}$$

we see that

$$\left| \frac{tz^2 f'(z)}{g(z)\ g(tz)} - 1 \right| \ \leq\ \frac{\sum_{n=2}^{\infty} |na_n - c_n|}{1 - \sum_{n=2}^{\infty} |c_n|}.$$

Therefore, if $f(z)$ satisfies (2.13), then we have (2.14). This completes the proof of Theorem 2.5 $\square$

3. Distortion and Covering Theorem

Theorem 3.1. *Let $f \in \mathcal{X}_t(\gamma)$. Then the unit disk $\mathbb{U}$ is mapped by $f(z)$ on a domain that contain the disk* $|w(z)| < \dfrac{1}{4-\gamma}$.

Proof. Suppose that $f(z) \in \mathcal{X}_t(\gamma)$, and let w_0 be any complex number such that $f(z) \neq w_0$ for $z \in \mathbb{U}$. Then $w_0 \neq 0$ and

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \cdots \tag{3.1}$$

is univalent in $\mathbb{U}$. This leads to

$$\left| a_2 + \frac{1}{w_0} \right| \leq 2, \tag{3.2}$$

on the other hand, from Corollary 2.1, we know that

$$|a_2| \leq 2 - \gamma, \;\; 0 \leq \gamma < 1. \tag{3.3}$$

combining (3.2) and (3.3), we deduce that

$$|w_0| \geq \frac{1}{|a_2| + 2} \geq \frac{1}{4-\gamma}. \tag{3.4}$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $f \in \mathcal{X}_t(\gamma)$, then we have*

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)^3} \quad (|z| = r, 0 \leq r < 1) \tag{3.5}$$

and

$$\int_0^r \frac{1-(1-2\gamma)\tau}{(1+\tau)^3} d\tau \leq |f(z)| \leq \int_0^r \frac{1+(1-2\gamma)\tau}{(1-\tau)^3} d\tau \quad (|z| = r, 0 \leq r < 1) \tag{3.6}$$

Proof. Suppose that $f(z) \in \mathcal{X}_t(\gamma)$. From the definition of subordination between analytic functions, we deduce that

$$\begin{aligned} \frac{1-(1-2\gamma)r}{1+r} &\leq \frac{1-(1-2\gamma)|w(z)|}{1+|w(z)|} \leq \left| \frac{tz^2 f'(z)}{g(z)g(tz)} \right| = \left| \frac{zf'(z)}{F(z)} \right| \\ &\leq \frac{1-(1-2\gamma)|w(z)|}{1+|w(z)|} \leq \frac{1+(1-2\gamma)r}{1-r} \quad (|z| = r, 0 \leq r < 1) \end{aligned} \tag{3.7}$$

where w is Schwarz function with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$. Since

$$F(z) = \frac{g(z)g(tz)}{tz}$$

is an starlike function, it is well known that [1]

$$\frac{r}{(1+r)^2} \leq |F(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r, 0 \leq r < 1). \tag{3.8}$$

Now it follows from (3.7) and (3.8), that

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)^3} \quad (|z| = r, 0 \leq r < 1).$$

Let $z = re^{i\theta}$ $(0 < r < 1)$. If $\mathcal{L}$ denotes that *closed* line segment in the complex ζ-plane from $\zeta = 0$ and $\zeta = z$, then we have

$$f(z) = \int_{\mathcal{L}} f'(\zeta)d\zeta = \int_0^r f'(\tau e^{i\theta})e^{i\theta}d\tau \quad (|z| = r, 0 \leq r < 1).$$

Thus by using upper estimate in (3.5), we have

$$|f(z)| = \left|\int_0^z f'(\zeta)d\zeta\right| \leq \int_0^r |f'(\tau e^{i\theta})|d\tau \leq \int_0^r \frac{1+(1-2\gamma)\tau}{(1-\tau)^3}d\tau \;\; (|z| = r, 0 \leq r < 1)),$$

which yields the right hand side of the inequality in (3.6). In order to prove the lower bound in (3.6), it is sufficient to show that it holds true for z_0 nearest to zero, where $|z_0| = r$ $(0 < r < 1)$. Moreover, we have

$$|f(z)| \geq |f(z_0)| \;\; (|z| = r, 0 \leq r < 1).$$

Since $f(z)$ is close-to-convex function in the open unit disk $\mathbb{U}$, it is univalent in $\mathbb{U}$. We deduce that the original image of the closed line segment $\mathcal{L}_0$ in the complex $\zeta-$plane from $\zeta = 0$ and $\zeta = f(z_0)$ is a piece of arc Γ in the disk $\mathbb{U}_r$, given by

$$\mathbb{U}_r = \{z : \; z \in \mathbb{C} \;\text{ and }\; |z| \leq r \; (0 \leq r < 1)\}.$$

Since, in accordance with (3.5), we have

$$|f(z)| = \int_{f(\Gamma)} |dw| = \int_{\Gamma} |f'(z)||dz| \geq \int_0^r \frac{1-(1-2\gamma)\tau}{(1+\tau)^3}d\tau \;\; (|z| = r, 0 \leq r < 1)).$$

This completes the proof of Theorem 3.2. □

4. Radius of Convexity

Theorem 4.1. *Let $f \in \mathcal{X}_t(\gamma)$, then $f(z)$ is convex in $|z_0| < r_0 = 2 - \sqrt{3}$.*

Proof. When $f(z) \in \mathcal{X}_t(\gamma)$, there exists $g(z) \in \mathcal{S}^*(1/2)$ such that (1.4) holds, then $F(z)$ defined by (2.2) is a starlike function, so from (1.4) we have

$$zf'(z) = F(z)p(z), \tag{4.1}$$

where $p(0) = 1$ and $\Re(p(z)) > 0$. From (4.1), we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zF'(z)}{F(z)} + \frac{zp'(z)}{p(z)},$$

so on using well know estimates [1], we have

$$\begin{aligned}\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &= \Re\left\{\frac{zF'(z)}{F(z)}\right\} + \Re\left\{\frac{zp'(z)}{p(z)}\right\} \\ &\geq \frac{1-r}{1+r} - \left|\frac{zp'(z)}{p(z)}\right| \\ &\geq \frac{1-r}{1+r} - \frac{2r}{1-r^2} = \frac{r^2-4r+1}{1-r^2}. \qquad (4.2)\end{aligned}$$

It is easily seen that, if $r^2 - 4r + 1 > 0$, then $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$. Let

$$H(r) = r^2 - 4r + 1, \tag{4.3}$$

since $H(0) = 1, H(1) = -2$, and $H'(r) = 2r - 4 < 0, \quad 0 \le r < 1$, this shows that $H(r)$ is monotonically decreasing function and thus equation $H(r) = r^2 - 4r + 1$ has a root r_0 in interval (0,1). On solving equation (4.3), we get $r_0 = 2 - \sqrt{3}$.

Thus when $r < r_0$, $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$, that is, $f(z)$ is convex in $|z| < r_0$. This completes the proof of Theorem 4.1. □

5. Fekete-Szego Inequality

In this section we assume that the function $\phi(z)$ is an univalent analytic function with positive real part that maps the unit disk $\mathbb{U}$ onto a starlike region which is symmetric with respect to real axis and is normalized by $\phi(0) = 1$, and $\phi'(0) > 0$. In such case , the function ϕ has an expression of the form $\phi(z) = 1 + B_1 z + B_2 z + ..., B_1 > 0$

Theorem 5.1. *for a function $f \in \mathcal{X}_t(\gamma)$,the following sharp estimate holds:*

$$|a_3 - \mu a_2^2| \le \frac{4}{3} + max\left\{\frac{2-2\gamma}{3}, |\frac{2\gamma}{3} + \mu\frac{\gamma^2}{3}|\right\} \quad (\mu \in \mathbb{C})$$

Proof. Since the function $f \in \mathcal{X}_t(\gamma)$,there is a normalized analytic function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$ such that

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \phi(z)$$

By using the definition(1.1) we find a function $w(z)$ analytic in U , normalized by $w(0) = 0$ satisfying $|w(z)| < 1$ and

$$\frac{tz^2 f'(z)}{g(z)g(tz)} = \phi(w(z)) \tag{5.1}$$

By writing $w(z) = w_1 z + w_2 z^2 + ...$ we see that

$$\phi(w(z)) = 1 - 2\gamma w_1 z + \left\{2(1-\gamma)w_2 - 2\gamma w_1^2)\right\} z^2 + ... \quad (5.2)$$

Also by g(z) given by (2.1)

$$\frac{g(z)g(tz)}{tz} = z + (b_2 + tb_2)z^2 + (b_3 + t^2 b_3 + b_2^2 t)z^3 + ...$$

and therefore

$$\frac{tz}{g(z)g(tz)} = \frac{1}{z} - (b_2 + b_2 t)z - (b_3 + b_3 t^2 + b_2^2 t)z^2 - ....$$

Using this and the Taylor's expansion for$zf'(z)$, we get

$$\frac{tz^2 f'(z)}{g(z)g(tz)} = 1 + 2a_2 z + (3a_3 - b_2 - b_2 t)z^2 + ... \tag{5.3}$$

using (5.1),(5.2)and (5.3)we get

$$2a_2 = -2\gamma w_1$$

$$3a_3 - b_2 - b_2 t = (2 - 2\gamma)w_2 - 2\gamma w_1^2$$

$$3a_3 = (1 + t)b_2 + (2 - 2\gamma)w_2 - 2\gamma w_1^2$$

this shows that

$$a_3 - \mu a_2^2 = \frac{1}{3}(1+t)b_2 + \frac{2-2\gamma}{3}\left\{w_2 - \left(\frac{2\gamma+\mu\gamma^2}{2-2\gamma}\right)\right\}$$

By using the following estimate ([7,inequality 7,p-10)

$$|w_2 - \lambda w_1^2| \leq max\left\{1, |\lambda|\right\}, \quad (\lambda \in \mathbb{C})$$

for an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ which is sharp for the function $w(z) = z^2$ or $w(z) = z$, the desired result follows upon using the estimate that $|1+t| \leq 2$, and $|b_2| \leq 2$.

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Department of Mathematics, Central University of Rajasthan, Kishangarh, Distt.-Ajmer, Rajasthan, India

E-mail address: jkp_0007@rediffmail.com

Department of Mathematics, Institute of Applied Sciences Humanities, G. L. A. University, Mathura, U. P., India,

E-mail address: ambuj_math@rediffmail.com