Gamma Distribution

Introduction

- The graph of any normal probability density function is the bell shaped and thus symmetric.
- There are many practical situations in which the variable of interest to the experimenter might have a skewed distribution.
- A family of p.d.f's that yield a wide variety of skewed distribution shape is the gamma family.
- The gamma distribution derives its name from the well-known gamma function, studied in many areas of mathematics.

Definition: The gamma function is defined by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \qquad \text{for} \quad \alpha > 0$$

After integration by parts, we obtain

$$\Gamma(\alpha) = -e^{-x}x^{\alpha-1}\Big|_0^\infty + \int_0^\infty (\alpha-1)e^{-x}x^{\alpha-2}dx$$

$$= (\alpha-1)\int_0^\infty e^{-x}x^{\alpha-2}dx$$

which yields the recursion formula

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$$

Repeated application of the recursion formula gives

$$\Gamma(\alpha) = (\alpha - 1)(\alpha - 2)\Gamma(\alpha - 2)$$

$$= (\alpha - 1)(\alpha - 2)(\alpha - 3)\Gamma(\alpha - 3)$$

When $\alpha = n$, and *n* is a positive integer,

$$\Gamma(n) = (n - 1)(n - 2)...\Gamma(1).$$

However, by the definition

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

and hence

$$\Gamma(n) = (n-1)!.$$

Definition: The continuous random variable *X* has a gamma distribution, with parameters α and β, if its density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Where $\alpha > 0$ and $\beta > 0$

- The standard gamma distribution has $\beta = 1$

- The mean and variance of the gamma distribution may be obtained by making use of the gamma function.

- For the mean we have

$$\mu = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{0}^{\infty} x \cdot x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \alpha\beta$$

- It can also be shown that $\sigma^2 = \alpha\beta^2$.

- The special gamma distribution for $\alpha = 1$ is called the exponential distribution.