Introduction to Probability Theory

Introduction

- Sample Spaces and Events
- Counting Techniques
- Algebra of Events
- Independence and Conditional Probability
- Total Probability and Baye’s Theorem

Sample Spaces and Events

- The term probability refers to the study of randomness and uncertainty.
- In any situation in which one of a number of possible outcomes may occur, the theory of probability provides methods for quantifying the chances or likelihood’s associated with the various outcomes in an experiment.

- An **experiment** is any action or process that generates observations.
- Each experiment terminates with an **outcome.**
- Example, when a coin is tossed, heads and tails occur at random. No one can yet predict whether the act of conception will produce a male or a female.

- **Definition** The sample space of an experiment, denoted by **S** is the set of all possible outcomes of that experiment.

In the outcome of an experiment, we usually identify the various outcomes in the sample space with numbers, points, or some other kinds of symbols.

Suppose that four students from UEW contested for the post of SRC President and we let *a, b, c*, and *d* denote the fact that it is won respectively by Mr. Aidoo, Mrs.Bawa, Ms Commey or Mr. Duckworth, then the sample space for this election (experiment) is the set

$$S = \{a, b, c, d\}$$

- In all the examples that we considered above we realise that a sample space can be looked upon as a universal set. That is all the possible outcomes in that experiment.
- Sample spaces are classified according to the number of elements (points) that they contain.

- In general, a sample space is said to be **discrete** if it has finitely many or a countable infinity of elements.
- If the elements of a sample space constitute a continuum, for example, all the points on a line, or all the points in a plane, the sample space is said to be **continuous**.

- In statistics, any subset of a sample space is called an **event***.*
- ***Definition*** An event denoted by A is any collection (subset) of outcomes contained in the sample space S. That is we write

$$A \subseteq S$$

- Each event has an associated quantity which characterises how likely its occurrence is.
- This is called the probability of the event.
- There are several approaches to the concept of probability.

- The frequency or statistical approach is based, on the concept of the frequency of an event in a long series of trials.
- The “classical” approach is to calculate the number of favourable outcomes of a trial and divide it by the total number of possible outcomes.

- If all possible outcomes of an experiment are assumed to be equally likely, then to determine the probability that a particular event *A* will occur, we need to count only two things
- All possible outcomes or the number of ways in which the experiment itself could proceed denoted by $n(S)$.
- The number of ways in which the event *A* could occur denoted by $n(A)$.

The probability that event *A* will occur is then given by

$$P(A) = \frac{n(A)}{n(S)}$$

Where $P(A)$ is read the probability of an event A

- Probabilities are numbers between 0 and 1 inclusive. They give us an idea of whether or not a physical event will occur. For ease in interpretation, these numbers can be expressed as percentages between 0% and 100%.

- Probabilities near 0 indicate that the event in question is not likely to occur. They do not mean that the events will not occur, only that it is considered to be rare.

- Probabilities near indicate that the event in question has about the same chance of occurring as it has of failing to occur.

- Probabilities near 1 indicate that the event in question is likely to occur. They do not mean that the event will absolutely occur, only that the event is considered to be a common occurrence.

Example 1.1

A box contains 24 batteries for use in radio sets. Four of these batteries are weak but this fact cannot be detected visually. If we select one battery from the box at random, what is the probability that the one chosen is weak?

Example 1.2

What is the probability of drawing an ace from a well-shuffled deck of 52 playing cards?

Example 1.3

In a certain village called Amanase, children use to play a game called *“Anhwe wekyire*” where a number of children are made to squat in a circular track facing the centre of the circle. One player runs around the circle holding a piece of cloth, which he/she places behind one of the children squatting without the person knowing it. If on a fine Monday evening 13 children play this game, what is the probability that the cloth will be placed behind Nana Adwoa ?

Counting Techniques

- Sometimes it can be quite difficult or at least tedious, to determine the number of elements in a finite set or subset by direct enumeration. The sample space can be constructed by listing the sample points.
- Thus, it is helpful to have a systematic method for determining or counting the elements of *S*.

Example 1.4

Suppose that Miss Joyce Amoako, a teacher in Accra, wishes to visit her mother at Hohoe. Suppose also that she has a choice of three different routes from Accra to Tema, a choice of two different routes from Tema to Akosombo and then a choice of three different routes from Akosombo to Hohoe. Find out how many possible routes Miss Amoako has at her disposal to travel from Accra to Hohoe through Tema and Akosombo.

- To solve problems of this nature, a configuration called a **tree diagram** can be used to represent pictorially all the possibilities.
- Following a given path from left to right along the branches of the tree, it can be seen that altogether there are 18 possibilities

- This result could also have been obtained by observing that there are three *T*-branches, that each *T*-branch forks into two *A*-branches and that each *A*-branch forks into three *H*-branches.
- There are, therefore, 3.2.3 = 18 combinations of branches or paths.

Example 1.5

A student wishes to select a language course from among English, French, Twi and Fante, a science course from among Biology, and Mathematics, and a Social Science course from among Psychology, Guidance and Counselling and Geography. In how many ways can the student set up a three-course schedule?

- By exploiting some general counting rules, it is possible to compute the number of ways an activity can take place.
- One of such rules is he product rule for ordered pairs.

- An ordered pair means that if A1 and A2 are objects then the pair (A1, A2) is different from the pair (A2, A1).

- For example, if an individual is travelling from Accra to Takoradi but wishes to make a stopover at Cape Coast, then he has four options of continuing his journey to Takoradi. He can do this by either the STC or GPRTU bus.
- The options available to travel from Accra to Takoradi via Cape Coast are therefore given by the ordered pair (STC, GPRTU), (GPRTU, STC), (STC, STC) or (GPRTU, GPRTU).

- **Proposition** If the first element or object of an ordered pair can be selected in n_1 ways and for each of these n_1 ways the second element of the pair can be selected in n_2, ways, then the number of pairs is n_1n_2 .
- This proposition can be generalised.

- **Multiplication (Product) Rule**

- If sets $A_1, A_2, \ldots A_k$ contain respectively n_1 $, n_2, \ldots n_k$ elements then, there are $n_1\, n_2 \ldots n_k$ ways of choosing first an element of A_1, then an element of A_2 …and finally an element of A_k.

In example 1.4, we had n_1 = 3, n_2 = 2 and n_3 = 3 and hence we had 3.2.3 = 18 possibilities. Verify that in example 1.5, n_1= 4, n_2 = 2, n_3 = 3. Therefore the number of possibilities are 4.2.3 = 24.

Example 1.6

In how many different ways can a class of 35 students choose a class prefect and his assistant?

Example 1.7

A game consists of tossing a coin and then throwing a die. How many equally likely outcomes do the game have? Draw a probability tree and list the possible outcomes.

Permutations

- The rule for the multiplication of choices is often used when several choices are made from one set and we are concerned with the order in which they are made.
- For example, if your surname is Tamakloe, then the order in which your name is spelt is very important.

- A permutation is an arrangement of objects in a definite order. Notice that the word permutation is the name given to an arrangement of objects, and that the key words in the definition are **order** and **arrangements.**

- **Definition If *r* objects are chosen from a set of *n* distinct objects, any particular arrangements or order of these objects is called a permutation.**

Example 1.8

Suppose that Swedru sports stadium has 5 gates. If a group of 5 boys are to enter the stadium, in how many ways can these boys be arranged at the gates?

Definition Let *n* be a positive integer. By *n* factorial, denoted by *n!*, we mean

$n! = n(n-1)(n-2)(n-3)\ldots 3.2.1$

Let us have *n* distinct objects from which we select *r* objects. The first selection is made from the whole set of *n* objects, the second selection is made from objects remaining and so on; and the *rth* selection is made from the objects

$$n-(r-1)=n-r+1$$ remaining.

Therefore, by the rule for the multiplication of choices, the total number of permutations of *r* objects selected from a set of *n* distinct objects denoted by ${}_nP_r$ and is given by

$${}_nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+1)$$

The use of factorial notation allows ${}_nP_r$ to be expressed more compactly. That is

$$ {}_nP_r = n(n-1)(n-2)(n-3)\ldots(n-r+1) $$

$$ = \frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)(n-r)(n-r-1)\ldots3.2.1}{(n-r)(n-r-1)\ldots3.2.1} $$

$$ = \frac{n!}{(n-1)!} $$

The number of permutations of *r* objects selected from a set of *n* distinct objects is therefore,

$$ {}_nP_r = n(n-1)(n-2)(n-3)\ldots(n-r+1) = \frac{n!}{(n-r)!} $$

Example 1.9

In how many different ways can one make a first, second and third choice among 38 Teacher Training Colleges in Ghana?

Permutations of Indistinguishable Objects

We have been assuming until now that the *n* objects from which we select *r* objects and form permutations are all distinct. Thus, the previous formula cannot be used when the objects are indistinguishable. Let *n* objects be divided into *k* categories with objects within categories being indistinguishable. The total number of distinguishable arrangements of these objects is given by

$$\frac{n!}{n_1!n_2!n_3!..n_k!}$$

Example 1.10

How many permutations are there of the letters of the word “redeemer”?

Combinations

- There are many problems in which we must find the number of ways in which *r* objects can be selected from a set of *n* objects, where the order in which the selection is made, is not important.

The number of ways in which **r** objects can be selected from a set of **n** distinct objects, without regard to order is called the number of combinations of **n** objects taken **r** at a time denoted by ${}^{n}C_{r}$ or $\binom{n}{r}$ That is

$$ {}^{n}C_{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} $$

Example 1.11

In how many different ways can 3 of 12 teaching assistants be chosen to assist with the marking of a mid-semester examination?

Example 1.12

A mathematics lecturer sets 6 questions in an end of semester examination and students are asked to attempt any 4 of these questions. In how many ways can the questions be selected?

Example 1.13

In how many different ways can the Director of a research laboratory choose two chemists from among seven applicants and three Physicists from among nine applicants?

Algebra of events

- When events *A* and *B* have no outcomes in common, they are said to be **mutually exclusive or disjoint.**
- Random events are said to form a complete group if in each experiment, any one of the events can occur but no disjoint event can occur.

In other words, $A_1, A_2, \ldots A_n$, form a complete group if $\sum_{i=1}^{n} A_i = S$. That is, at least one of the events is certain to occur as a result of an experiment.

Example 1.15

An experiment consists in throwing a die.

The events $A = \{1,2\}$ $B = \{2,3,4\}$ and $C = \{4,5,6\}$

form a complete group since

$$A + B + C = \{1,2,3,4,5,6\} = S$$

Axioms of Probability Theory

- The probability of an event *A* is a non negative real number, that is $P(A) \geq 0$ for any event A in the sample space S.
- For certain events $P(S) = 1$.
- If A_1, A_2, are mutually exclusive events of the sample space S then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

The following deductions can be made from the axioms.

- For any event *A*, $P(A)=1-P(A')$,where is the A' complement of *A*.
- No matter what the random event, *A* is,

$$0 \subseteq P(A) \subseteq 1$$

- If events $A_1, A_2 \ldots A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \ldots A_n) = P(A_1) + P(A_2) + \ldots P(A_n)$$

or in the compact form we write as

$$P\left(\sum^{n} A_i\right) = \sum^{n} P(A_i)$$

Example 1.16

In a sporting game, shots are fired at a certain domain D, consisting of three non-overlapping zones. The probability of hitting zone 1 is 5%. The probabilities of hitting zones II and III are 10% and 17% respectively. What is the probability of hitting D?

- Two events are called **complementary** if they are mutually exclusive and form a complete group.
- If one event is denoted by A, its complement is denoted by A' .
- Let the probability of the occurrence of events A and A' be *p* and *q* respectively.

- From the axioms we have: $P(A+A') = P(S) = 1$
- This implies that $P(A+A') = P(A) + P(A') = p + q = 1$
- We deduce from this statement that if random events $A_1 A_2 \ldots A_n$ form a complete group of exclusive events then

$$P(A_1) + P(A_2) + \ldots + P(A_n) = 1$$

Random events A and B are called **compatible** if in a given trial both events can occur which is to say we have a logical product. That is

$$P(A \cap B) = P(A) \cdot P(B)$$

Theorem If two events A and B are compatible then

$$P(A+B)=P(A)+P(B)-P(AB).$$

Example 1.17

Events A and B are such that

$$P(A)=\frac{1}{5},\ P(B)=\frac{1}{6}, P(A \text{ and } C)=\frac{1}{20} \quad \text{and} \quad P(B \text{ or } C)=\frac{3}{8}$$

Evaluate

$$P(C)$$

$$P(B \cap C)$$

$$P(B)P(C)$$

Example 1.18

In a certain school, a group of children take examination in mathematics and English. The proportion passing is 80% and 70% respectively, while 10% fail both examinations. Find the probability that a child passes both examinations.

Independence and Conditional Probability

Example 1.19

A box contains 6 blue and 4 red pens. Two pens are picked at random in succession. If a blue pen was selected first, can the second pen be blue also?

- An event *A* is said to be independent of an event *B*, if the probability of occurrence of *A* does not depend on whether event *B* took place or not.

- **Definition** Two events A and B are said to be independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Example 1.20

Two cards are drawn at random from an ordinary deck of 52 playing cards. What is the probability of getting two aces if:

1. the first card is replaced before the second card is drawn
2. the first card is not replaced before the second card is drawn

- **Definition** Events $A_1, A_2, \cdots A_n$ are mutually independent if

$$P(A_1 \cap A_2 \ldots \cap A_n) = P(A_1) \cdot P(A_2) \ldots P(A_n).$$

- Thus *n* events are mutually independent if the probability of the intersection of any subset of the *n* events is equal to the product of the individual probabilities.

- In example 1.17 show that that events *B* and *C* are independent.

- **Definition** The probability that event *A* occurred, provided that event *B* took place will be denoted by $P(A/B)$ and will be called the conditional probability of event *A* provided that *B* has occurred.

Example 1.22 A box contains 3 red and 2 black pens. A pen is drawn and then a second is drawn. Let B be the event of the occurrence of a red pen in the first drawing and event A, the occurrence of a red pen in the second drawing. Show that

$$P(A/B) = \frac{2}{4} = \frac{1}{2}$$

Definition The probability of the products of two events *A* and *B* is equal to the probability of one of them (say *A*) multiplied by the conditional probability of the other, provided that the first event has occurred. Mathematically, we write

$$P(AB) = P(A)P(B/A) \quad \text{or} \quad P(BA) = P(B)P(A/B)$$

After rearrangement we get

$$P(A/B) = \frac{P(AB)}{P(B)} \quad , \qquad P(B) \neq 0$$

Example 1.23

Suppose that at a Mobil filling station, 10% of drivers check their oil level, 20% check their tyre pressure and 1% check both oil and tyre pressure. Suppose also that a driver is selected in a random fashion. What is the probability that if a driver checked his or her tyre pressure, the oil level was also checked?

This rule can be generalised to an arbitrary number of events. If we have n events, $A_1 A_2 \ldots A_n$, then

$$P(A_1 A_2 \ldots A_n) = P(A_1)P(A_2 / A_1)P(A_3 / A_1 A_2)\ldots P(A_n / A_1 A_2 \ldots A_{n-1})$$

Total Probability and Baye's Theorem

- In the last section we studied conditional probability that yielded the multiplication rule. succession.

- The general multiplication rule is useful in solving many problems in which the ultimate outcome of an experiment depends on the outcome of various intermediate stages.

Theorem If event *A* can be realised only when one of the events $B_1, B_2, \cdots B_n$ which form a complete group of exclusive events, occur, then the probability of event *A* is calculated from the formula:

$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \ldots + P(B_n)P(A/B_n)$$

or compactly

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

Proof Event *A* can occur if one of the compatible events (B_1 and A), (B_2 and A) … (B_n and A) is realised. Consequently, by the theorem of addition of probabilities we get:

$$P(A) = P(B_1 \; and \, A) + P(B_2 \; and \, A) + \ldots P(B_n \; and \, A)$$

$$= P(B_1 \cap A) + P(B_2 \cap A) + \ldots + P(B_n \cap A)$$

$$= P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \ldots P(B_n)P(A/B_n)$$

$$= \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

This implies that if the conditions of the theorem are fulfilled then we find that

$$P(A) = \sum_{i=1}^{n} P(B_i) P(A/B_i)$$

This is the formula for total probability.

Example 1.24

A chain of video stores in Accra Central sell three different brands of video cassette recorders (VCR's). Fifty percent of their products are Panasonic. 30% are Sony and 20% are Goldstar brands. Each manufacturer offers a one-year warranty on parts and labour. It is known that 25% of Panasonic brand require warranty repair work, whereas the corresponding percentages for Sony and Goldstar brands are 20% and 10% respectively. What is the probability that a randomly selected purchaser has bought:

What is the probability that a randomly selected purchaser has bought:

1. a Panasonic VCR that will need repair while under warranty?
2. a Sony VCR that will need repair while under warranty?
3. a Goldstar VCR that will need repair while under warranty?
4. Find the probability that a randomly selected purchaser has a VCR that will need repair while under warranty.

Example 1.25

Suppose we have three identical boxes. The first box contains a white and b black balls. The second box contains c white and d black balls and the third box contains only white balls. A ball is drawn at random from one of the boxes. Find the probability that the ball is white.

Bayes', Theorem

- Bayes' theorem provides a formula for finding the probability that the "effect" *A* was "caused" by the event B_n. `
- We consider a complete group of exclusive events, $B_{1,} B_2 \ldots B_n$, the probability of occurrence of which are $P(B_1)$, $P(B_2)$, …$P(B_n)$. Event *A* can occur only together with one of the events $B_1 B_2 \ldots B_n$, which we shall call causes.

Suppose that event *A* has taken place. This will alter the probability of the causes $P(B_1),P(B_2), \ldots P(B_n)$. We are required to determine the condition of these causes on the assumption that event A has occurred.

That is we are to determine

$$P(B_1 / A), P(B_2 / A), \ldots P(B_n / A)$$

Let us determine the probability *P(A* and *B*$_1$*)*

$P(A \text{ and } B_1)=P(AB)=P(A)P(B_1/A)$.

This implies that

$$P(B_1/A)=\frac{P(AB_1)}{P(A)}=\frac{P(B_1)P(A/B_1)}{P(A)}$$

Substituting for *P(A)* from the total probability formula, we get

$$P(B_1/A) = \frac{P(B_1)P(A/B_1)}{P(A)} = \frac{P(B_1)P(A/B_1)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}$$

The probability $P(B_2/A)$, $P(B_3/A)$, …$P(B_n)$ are determine in similar fashions. In general we have

$$P(B_k / A) = \frac{P(B_k)P(A/B_k)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}$$

k=1,2,3,…,n, i=1,2,…,n

Bayes' theorem Let events $B_1 B_2 \ldots B_n$ be a collection of n mutually exclusive events for *i=1,2,…n*. Then for any other event A for which P(A)>0,

$$P(B_k / A) = \frac{P(B_k)P(A/B_k)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}$$

k=1,2,3,…,n i=1,2,…,n.

Example 1.26

With reference to example 1.24, if a customer returns to the store with a VCR that needs warranty repair work, what is the probability that it is a

- Panasonic brand?
- Sony brand?
- Goldstar brand?

Example 1.27

With reference to example 1.25, find the probability that the ball came from

- the first box.
- the second box.
- the third box.