Fundamental Concepts of ODE

Differential Equations ?

- The world we live in is of interrelated changing entities.

- The position of a moving object changes with time

- The bending of a beam changes with the load placed on it, and many more.

DE?

- In the field of mathematics, these changing entities are called **variables** which can be linked together.
- A linkage or a relation, in which one variable, say *y*, depends according to a certain law on another variable say *x*, defines a function.
- We write $y = f(x)$ for this relation.

DE?

- The rate of change of the dependent variable *y*, with respect to the independent variable *x*, is the derivative of the function, denoted by y' or $\frac{dy}{dx}$.

- The variables and the function together with its derivatives can also be linked together by an equation.

DE

A differential equation is an equation involving the unknown function

$$y = f(x),$$

together with its derivatives

$$y', y'', \ldots, y^{(n)}.$$

Mathematically, a differential equation may be written as follows:

Implicit form $$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

Explicit form $$y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

Examples of DE

$$y' + xy = x^2$$

$$(y')^3 = \sin x$$

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$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$\frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} + y = e^x$$

Classification of DE's

1. **Ordinary Differential Equation (ODE)**

   When the unknown function appearing in an equation has only a single independent variable.

2. **Partial Differential Equation (PDE)**

   When the unknown function depends upon more than one independent variable.

Order of DE's

The order of a DE is the order of the highest derivative of the unknown function that appears in the equation.

In general a differential equation of order *n* can be written as:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-i)} + ... + a_1(x)y' + a_0(x)y = F(x)$$

Linear DE’s

The ordinary differential equation

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

is said to be linear if:

- the dependent variable and its derivatives occur to the first degree only.
- no products of *y* and/or any of its derivatives are present and.
- no transcendental functions of *y* and/or its derivatives occur.

Non-Linear DE's

An ordinary differential equation is said to be **non-linear** if it is not linear i.e. if any of conditions above (1) –(3) is violated.

Examples

$$y'' + 5y' + 6y = 0$$

$$y^{(4)} + x^2 y^{(3)} + x^3 y' = xe^x$$

$$y'' + 5y' + 6y^2 = 0$$

$$y'' + 3(y')^2 + \sin xy = 0$$

$$y'' + 4y(y') + 6y = 0$$

Solution of DE’s

The solution or integral of a differential equation is any function $y = \varphi(x)$, which when put into the equation, gives an identity (i.e. satisfies the equation).

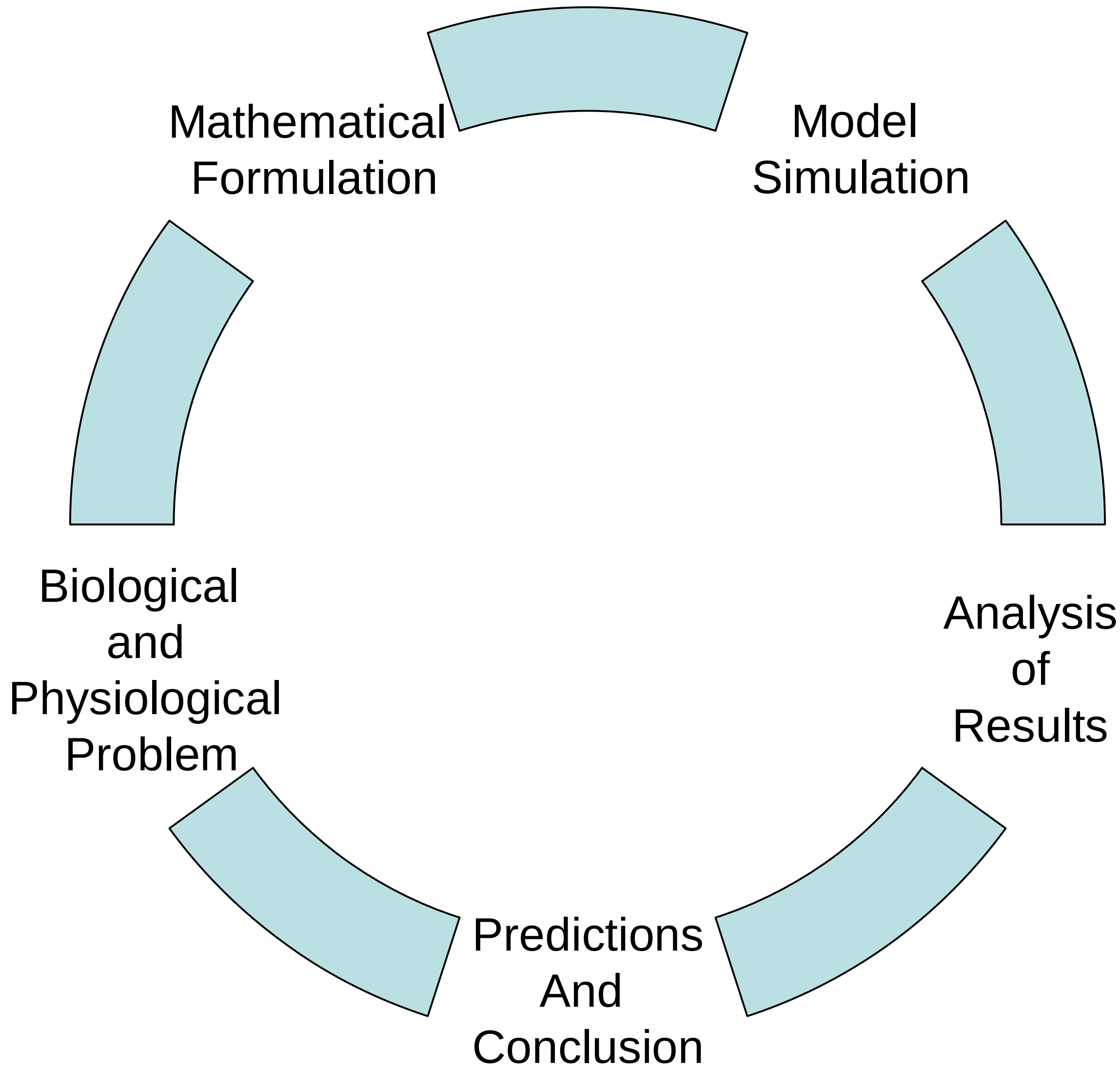
For $y'' + y = 0$

Solutions $\varphi(x) = y = \sin x$

$\varphi(x) = y = 2\cos x$

Mathematical Modelling

- The main reason for solving many differential equations is to try to learn something about an underlying physical process that the equation is believed to model.
- In the modelling process, we are concerned with more than just solving a particular problem. The complete solution process consists of the following steps.


Mathematical Formulation
Model Simulation
Analysis of Results
Predictions And Conclusion
Biological and Physiological Problem

First Order Differential Equations

- Differential equations of the first order arise in a variety of problems.

- The types of differential equations involved in these applications fall into several classifications, each of which demands a different method of solution.

These include the determination of the velocity of free-falling bodies subject to a resistive force; finding curves of population growth, radioactive decay and the pursuit of a predator tracking its prey and finding the current or change in an electrical circuit.

Differential equations of the first order is written in the form

$$y' = \frac{dy}{dx} = f(x, y)$$

where f is a given function of two variables.

Geometric Interpretations

For each point m with co-ordinates *x* and *y*, this equation defines the value of the derivative y' or the **slope** of the tangent line to the integral curve passing through this point.

Consequently, from the geometric point of view, the problem of integrating a differential equation consists in finding curves, the directions of the tangents which coincide with the direction of the field at the corresponding points.

Application of graphic calculus

Solutions of First Order Equations

The general solution of a first order differential equation is a function $y = \phi(x,c)$ that depends on a single arbitrary constant and satisfies the following conditions:

- For any specific value of the constant c, the equation holds true.

- For any point *x* in the interval of definition *I*, the point $(x_0, y(x_0))$ should lie in the interval of the domain.

Example 2.1

Verify that $y = cx^2$ is a solution to $xy' = 2y$.

Example 2.2

Verify that the function $y = x^2 + cx$ is a general solution to the differential equation $y'x - x^2 - y = 0$ where c is an arbitrary

- A particular solution to a first order differential equation $y' = f(x, y)$ is any function $y = \phi(x, c_0)$ which is obtained from the general solution $y = \phi(x, c)$ if in the latter we assign to the arbitrary constant a definite value $c = c_0$.

- The relation $y = \phi(x, c_0)$ is called a particular solution to the differential equation $y' = f(x, y)$.

Example 2.3

Find the particular solution of the equation

$y' = -\frac{y}{x}$ that satisfies the condition $x = 2$ when,

$y = 1$ if the general solution is given by the function

$$y = \frac{c}{x}$$

- From the geometric point of view, the general solution of a first order differential equation is a family of curves in a co-ordinate plane, which depends on a single arbitrary constant .
- These curves are referred to as **integral curves.** A particular integral is associated with one curve of the family that passes through a certain given point of the plane.

Initial Value Problems (IVP)

- In most applications the unknown function *y* must satisfy certain restraints, or auxiliary conditions in addition to satisfying the differential equation.

- These conditions imposed upon the problem, determine which of an infinite collection of solutions are peculiar to the given problem.

- The number of these conditions is usually equal to the order of the equation.

- When the auxiliary conditions are all specified at a single value of x, we refer to the problem as **initial value problem (IVP**).

- If the auxiliary conditions are specified at more than one point on the interval of interest, the resulting problem is called a **boundary value problem (BVP)**.

An IVP can be defined as a system consisting of the differential equation and the initial conditions:

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

The following are two examples of initial value problems:

$$\begin{cases} y' - y = 0 \\ y(0) = 2 \end{cases}$$

$$\begin{cases} y'' + y = 0 \\ y(1) = 2 \\ y'(1) = 0 \end{cases}$$

Theorem of Existence and uniqueness

If f and $\dfrac{\partial f}{\partial y}$ are both continuous functions in

the same domain of the *x*-*y*- plane containing the point, $(x_0,\ y_0)$
then there exists a unique solution of the IVP

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

defined on some interval , $|x - x_0| \leq h$ where *h* is

“sufficiently small”.

Example 2.4

Show that the given IVP has a unique solution in some interval of the form $-h \leq x \leq h$.

$$\begin{cases} y' = x^2 + y^2 \\ y(0) = 0 \end{cases}$$

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Example: Given the IVP, $\frac{dy}{dx} = y^{\frac{1}{3}}, \qquad y(0) = 0$

determine the existence and uniqueness of this problem. Verify also that the function

$$y = \left(\frac{2x}{3} \right)^{\frac{3}{2}}, x \geq 0$$

is a solution to this problem. Can you think of another solution?