A Study of Non-Canonical Lagrangian With ϕ^6 Potential

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Abstract

In recent years, the non-canonical lagrangian have been extensively studied in the context of inflationary cosmological models. In this work, the non-canonical Lagrangian with ϕ^6 potential have been discussed. We have obtained the periodic and solitary wave solutions of non-canonical Lagrangian with ϕ^6 potential using the auxiliary equation method. The solution is obtained in terms of elliptic and hyperbolic functions. In the theory of phase transition, the ϕ^6 potential, $V(\phi) = \frac{a}{2}\phi^2 - \frac{b}{4}\phi^4 + \frac{c}{6}\phi^6$, describes a first order phase transition. This potential is also called as Landau free energy density. We have also included a spatial gradient term (also called as Ginzburg term) in the non-canonical Lagrangian density. The solutions obtained here may provide some new direction in the theory of phase transition, field theory and related phenomena.

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I. INTRODUCTION

In recent years, non canonical Lagrangian have been used in various cosmological models [1]. The non-canonical Lagrangian is the Lagrangian with non-quadratic kinetic energy term. The mathematical form of canonical Lagrangian density is given by

$$\mathcal{L} \;=\; \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) - V(\phi),$$

where $V(\phi)$ is the potential energy and $\mu = 0,\ 1$. Here $\mu = 0$ corresponds to time coordinate t, and $\mu = 1$ corresponds to space coordinate x. Furthermore, the metric tensor $g^{\mu\nu}$ is diagonal, with $g^{00} = -g^{11} = 1$. Also, we are using natural units in which $c = 1$ and action is dimensionless. In this framework, the canonical Lagrangian density can be written as

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right] - V(\phi)$$

Note that the kinetic term $\frac{\partial\phi}{\partial t}$ is quadratic. On the other hand, the mathematical form of non-canonical Lagrangian density is written as

$$\mathcal{L} \;=\; k\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right)^n - V(\phi), \tag{1}$$

where $V(\phi)$ is the potential, n is an integer and k is a constant. For $n = 1$, we get the canonical Lagrangian of the field theory. Also, note that the Eq. (1) is Lorentz invariant for any value of n. The non canonical Lagrangian density have been studied in the inflationary cosmological models [1]. In Ref. [1], it has been shown that a non-canonical Lagrangian density (Lagrangian with non-quadratic kinetic term) without the potential energy can derive an inflationary evolution and in [4], it has been shown that the non-canonical scalars can significantly improve the viability of inflationary models. The solitary wave solution of Eq. (1) for certain class of potential have been obtained in [5]. The purpose of this work is to obtain the exact solution of Eq. (1) in 1+1 dimension for $n = 2$, and for the potential

$$V(\phi) \;=\; \frac{a}{2}\phi^2 - \frac{b}{4}\phi^4 + \frac{c}{6}\phi^6, \tag{2}$$

where a, b and c are real constants. This potential is useful in certain crystalline phase transition [2]. This potential (also known as Landau free energy density) have been used to study the formation of static domain walls [2, 3] in the theory of continuous phase transition. For $n = 2$, Eq. (1) takes the form

$$\mathcal{L} \;=\; \frac{k}{4}\left(\partial_\mu\phi\partial^\mu\phi\right)^2 - \left(\frac{a}{2}\phi^2 - \frac{b}{4}\phi^4 + \frac{c}{6}\phi^6\right). \tag{3}$$

Simplifying Eq. (3), we obtain

$$\mathcal{L} = \frac{k}{4}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right]^2 - \left(\frac{a}{2}\phi^2 - \frac{b}{4}\phi^4 + \frac{c}{6}\phi^6\right). \tag{4}$$

Note that the kinetic term is non-quadratic. The corresponding equation of motion is given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

Thus the equation of motion which follows from Eq. (4) is given by

$$3k\left(\frac{\partial\phi}{\partial t}\right)^2\left(\frac{\partial^2\phi}{\partial t^2}\right) - k\left(\frac{\partial^2\phi}{\partial t^2}\right)\left(\frac{\partial\phi}{\partial x}\right)^2 + 3k\left(\frac{\partial^2\phi}{\partial x^2}\right)\left(\frac{\partial\phi}{\partial x}\right)^2 - k\left(\frac{\partial\phi}{\partial t}\right)^2\left(\frac{\partial^2\phi}{\partial x^2}\right) + \frac{\partial V}{\partial\phi} = 0$$

From here one can see that the equation of motion remains second order [4]. Since Eq. (3) is Lorentz invariant, we can solve this equation for time independent case. Note that this is true only for wave solution. Thus the static equation of motion which follow from Eq. (4) is

$$3k\left(\frac{\partial\phi}{\partial x}\right)^2\left(\frac{\partial^2\phi}{\partial x^2}\right) + a\phi - b\phi^3 + c\phi^5 = 0. \tag{5}$$

Now, to find the exact solution of Eq. (5), we employ the method of auxiliary equation method [6,7]. According to this method, we make an ansatz for the solution of Eq. (5) as

$$\phi(x) = \sum_{i=0}^{l} a_i z^i(x), \tag{6}$$

where a_i $(i = 0, 1, 2, ..., l)$ are all real constants to be determined, l is a positive integer which can be determined by balancing the highest order derivative terms with the highest power nonlinear terms in Eq. (5) and $z(x)$ satisfies the following new auxiliary ordinary differential equation, namely

$$\left(\frac{dz}{dx}\right)^2 = Az^4(x) + Bz^3(x) + Cz^2(x) + D, \tag{7}$$

in which A, B, C and D are real constants. Using the balancing procedure in Eq. (5) one obtains $l = 2$, which in turn leads to the choice of $\phi(x)$ in Eq. (6) as

$$\phi(x) = a_0 + a_1 z(x) + a_2 z^2(x). \tag{8}$$

Substituting (8) along with (7) into Eq. (5) and then setting the coefficients of $z^j(x)$ ($j = 0, 1, ..., 10$), to zero in the resultant expression, one obtains a set of algebraic equations involving a_0, a_1, a_2, a, b, c, A, B, C and D as

$$6ka_1^2a_2D^2 + aa_0 - ba_0^3 + ca_0^5 = 0, \tag{9}$$

$$3ka_1^3CD + 24ka_1a_2^2D^2 - 3ba_0^2a_1 + 5ca_0^4a_1 + aa_1 = 0, \tag{10}$$

$$\frac{9}{2}kBDa_1^3 + 30kCDa_1^2a_2 + 24kD^2a_2^3 + aa_2 - 3ba_0a_1^2 - 3ba_0^2a_2 + 10ca_0^3a_1^2 + 5ca_0^4a_2 = 0, \tag{11}$$

$$3kC^2a_1^3 + 6kADa_1^3 + 39kBDa_1^2a_2 + 84kCDa_1a_2^2 - ba_1^3 - 6ba_0a_1a_2 + 10ca_0^2a_1^3 + 20ca_0^3a_1a_2 = 0, \tag{12}$$

$$\frac{15}{2}kBCa_1^3 + 24ka_1^2a_2C^2 + 48kADa_1^2a_2 + 102kBDa_1a_2^2 + 72kCDa_2^3 - 3ba_1^2a_2 - 3ba_0a_2^2 + 5ca_0a_1^4 + 30ca_0^2a_1^2a_2 + 10ca_0^3a_2^2 = 0, \tag{13}$$

$$\frac{9}{2}kB^2a_1^3 + 9kACa_1^3 + 57kBCa_1^2a_2 + 60kC^2a_1a_2^2 + 120kADa_1a_2^2 + 84kBDa_2^3 - 3ba_1a_2^2 + 20ca_0a_1^3a_2 + 30ca_0^2a_1a_2^2 + ca_1^5 = 0, \tag{14}$$

$$\frac{21}{2}kABa_1^3 + 33kB^2a_1^2a_2 + 66kACa_1^2a_2 + 138kBCa_1a_2^2 + 48kC^2a_2^3 + 96kADa_2^3 - ba_2^3 + 5ca_1^4a_2 + 30ca_0a_1^2a_2^2 + 10ca_0^2a_2^3 = 0, \tag{15}$$

$$6kA^2a_1^3 + 75kABa_1^2a_2 + 156kACa_1a_2^2 + 78kB^2a_1a_2^2 + 108kBCa_2^3 + 10ca_1^3a_2^2 + 20ca_0a_1a_2^3 = 0, \tag{16}$$

$$42ka_1^2a_2A^2 + 174kABa_1a_2^2 + 60kB^2a_2^3 + 120kACa_2^3 + 10ca_1^2a_2^3 + 5ca_0a_2^4 = 0, \tag{17}$$

$$96ka_1a_2^2A^2 + 132kABa_2^3 + 5ca_1a_2^4 = 0, \tag{18}$$

$$72ka_2^3A^2 + ca_2^5 = 0. \tag{19}$$

Now, we shall solve set of Eqs. (19) for various choices of constants A, B, C and D appearing in Eq. (7).

Case (1a): If we take $D = 0$ and $B = 0$ in the set of Eqs. (9-19), then one obtains the trivial solution, i.e., $\phi(x, t) = 0$ for all x and t. Next we discuss the special cases of Eq. (7) when $B = 0$ and $D \neq 0$. Let us solve Eq. (19) to obtain

$$a_2 = \pm\sqrt{\frac{-72k}{c}}\ A \tag{20}$$

Next we solve Eq. (18) to obtain a_1 as

$$a_1 = \frac{1}{2}\frac{B}{A}a_2 \tag{21}$$

For B=0, we obtain $a_1 = 0$. Next we solve Eq. (17) for a_0 to obtain

$$a_0 = \mp\sqrt{\frac{-8k}{c}}\ C \tag{22}$$

Solving Eq. (16), we obtain trivial solution, i.e., 0=0. Then solving Eq. (15) to obtain

$$b = k(96AD - 32C^2) \tag{23}$$

Then Eq. (14) yields 0=0 and from Eq. (13), we obtain

$$72kCDa_2 - 3ba_0 - 80kC^2a_0 = 0 \tag{24}$$

Equation (12) yields 0=0 and from Eq. (11), we obtain

$$ac = 1728k^2A^2D^2 + 448k^2C^4 - 2304k^2ADC^2 \tag{25}$$

Next, Eq. (10) yields 0=0 and from Eq. (9), we obtain

$$a = ba_0^2 + 8kC^2a_0^2 \tag{26}$$

Let us now simplify Eq. (24) using Eqs. (21), (22) and (23). Solving these set of equations, we obtain

$$C^2 = \frac{2016}{64}\ AD = 31.5\ AD$$

Using this value of C^2 in Eq. (23), we obtain

$$b = -912\ kAD \tag{27}$$

From Eq. (26), we obtain the constraining relation among the parameters as

$$ac = 166320\ k^2A^2D^2 \tag{28}$$

and from Eq. (25), we obtain

$$ac = 373680\ k^2A^2D^2$$

Thus we get two values of ac and hence the set of Eqs. (9)-(19) are inconsistent. Next, we add a gradient term in the Lagrangian. Thus we write Eq. (3) as

$$\mathcal{L} = \frac{k}{4}\left(\partial_\mu\phi\ \partial^\mu\phi\right)^2 + \psi\left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{a}{2}\phi^2 - \frac{b}{4}\phi^4 + \frac{c}{6}\phi^6\right) \tag{29}$$

The presence of second term in this equation violates the Lorentz invariance of the equation. Note that the conventional Lagrangian with gradient term have been studied in the context of field theory and cosmology. In crystallography, the coefficient ψ is known as Ginzburg term [2]. The solution of the Eq. (29) without the kinetic term has been obtained in Ref. [2]. The above Lagrangian density in the presence of Ginzburg term can be written as

$$\mathcal{L} = \frac{k}{4}\left[\left(\frac{\partial\phi}{\partial t}\right)^4 + \left(\frac{\partial\phi}{\partial x}\right)^4 - 2\left(\frac{\partial\phi}{\partial t}\right)^2\left(\frac{\partial\phi}{\partial x}\right)^2\right] + \psi\left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{a}{2}\phi^2 - \frac{b}{4}\phi^4 + \frac{c}{6}\phi^6\right)$$

The corresponding equation of motion is given by

$$\begin{aligned} 3k\left(\frac{\partial\phi}{\partial t}\right)^2\left(\frac{\partial^2\phi}{\partial t^2}\right) - k\left(\frac{\partial\phi}{\partial x}\right)^2\left(\frac{\partial^2\phi}{\partial t^2}\right) + 3k\left(\frac{\partial\phi}{\partial x}\right)^2\left(\frac{\partial^2\phi}{\partial x^2}\right) - \\ k\left(\frac{\partial\phi}{\partial t}\right)^2\left(\frac{\partial^2\phi}{\partial x^2}\right) + 2\psi\left(\frac{\partial^2\phi}{\partial x^2}\right) + a\phi - b\phi^3 + c\phi^5 \;=\; 0 \end{aligned} \tag{30}$$

Again, we solve Eq. (29) using auxiliary equation method. We take Eq. (6) as an ansatz for the solution of Eq. (29). Using the balancing procedure in Eq. (29) one obtains $l = 2$, which in turn leads to the choice of $\phi(x)$ in Eq. (6) as Eq. (8). Since the Lagrangian density (29) is not Lorentz invariant, this time we have to take into account the time derivative in equation of motion (30). Let us first make a change of variable, by defining $\xi = x - wt$. Using this transformation into Eq. (30), we get

$$k(3w^4\phi'^2\phi'' - w^2\phi'^2\phi'' + 3\phi'^2\phi'' - w^2\phi'^2\phi'') + 2\psi\phi'' + a\phi - b\phi^3 + c\phi^5 = 0$$

which can be written as

$$k(3w^4\phi'^2\phi'' - 2w^2\phi'^2\phi'' + 3\phi'^2\phi'') + 2\psi\phi'' + a\phi - b\phi^3 + c\phi^5 = 0$$

or,

$$k\phi'^2\phi''(3w^4 - 2w^2 + 3) + 2\psi\phi'' + a\phi - b\phi^3 + c\phi^5 = 0$$

where

$$\phi' = \frac{\partial\phi}{\partial\xi} \qquad \text{and} \qquad \phi'' = \frac{\partial^2\phi}{\partial\xi^2}$$

Again, Substituting (8) along with (7) into Eq. (30) and then setting the coefficients of $z^j(x)(j = 0, 1, \ldots, 10)$, to zero in the resultant expression, one obtains a set of algebraic equations (similar to Eqs. (9)-(19)) involving $a_0,\ a_1,\ a_2,\ a,\ b,\ c,\ A,\ B,\ C$ and D as

$$(3w^4-2w^2+3)(2ka_1^2a_2D^2)+4D\psi a_2+aa_0-ba_0^3+ca_0^5 \;=\; 0, \tag{31}$$

$$(3w^4-2w^2+3)[ka_1^3CD+8ka_1a_2^2D^2]+2\psi Ca_1-3ba_0^2a_1+5ca_0^4a_1+aa_1 \;=\; 0, \tag{32}$$

$$\begin{aligned}(3w^4-2w^2+3)\left[\frac{3}{2}kBDa_1^3+10kCDa_1^2a_2+8kD^2a_2^3\right]+&\\ 3B\psi a_1+8C\psi a_2+aa_2-3ba_0a_1^2-3ba_0^2a_2+10ca_0^3a_1^2+5ca_0^4a_2 \;=\;& 0,\end{aligned} \tag{33}$$

$$\begin{aligned}(3w^4-2w^2+3)[kC^2a_1^3+2kADa_1^3+13kBDa_1^2a_2+28kCDa_1a_2^2]+&\\ 4A\psi a_1+10B\psi a_2-ba_1^3-6ba_0a_1a_2+10ca_0^2a_1^3+20ca_0^3a_1a_2 \;=\;& 0,\end{aligned} \tag{34}$$

$$\begin{aligned}(3w^4-2w^2+3)\left[\frac{5}{2}kBCa_1^3+8ka_1^2a_2C^2+16kADa_1^2a_2+34kBDa_1a_2^2+\right.&\\ \left.24kCDa_2^3\right]+12A\psi a_2-3ba_1^2a_2-3ba_0a_2^2+5ca_0a_1^4+30ca_0^2a_1^2a_2+10ca_0^3a_2^2 \;=\;& 0,\end{aligned} \tag{35}$$

$$\begin{aligned}(3w^4-2w^2+3)\left[\frac{3}{2}kB^2a_1^3+3kACa_1^3+19kBCa_1^2a_2+20kC^2a_1a_2^2+\right.&\\ \left.40kADa_1a_2^2+28kBDa_2^3\right]-3ba_1a_2^2+20ca_0a_1^3a_2+30ca_0^2a_1a_2^2+ca_1^5 \;=\;& 0,\end{aligned} \tag{36}$$

$$\begin{aligned}(3w^4-2w^2+3)\left[\frac{7}{2}kABa_1^3+11kB^2a_1^2a_2+22kACa_1^2a_2+46kBCa_1a_2^2+\right.&\\ \left.16kC^2a_2^3+32kADa_2^3\right]-ba_2^3+5ca_1^4a_2+30ca_0a_1^2a_2^2+10ca_0^2a_2^3 \;=\;& 0,\end{aligned} \tag{37}$$

$$\begin{aligned}(3w^4-2w^2+3)[2kA^2a_1^3+25kABa_1^2a_2+52kACa_1a_2^2+26kB^2a_1a_2^2+&\\ 36kBCa_2^3]+10ca_1^3a_2^2+20ca_0a_1a_2^3 \;=\;& 0,\end{aligned} \tag{38}$$

$$\begin{aligned}(3w^4-2w^2+3)[14ka_1^2a_2A^2+58kABa_1a_2^2+20kB^2a_2^3+40kACa_2^3]+&\\ 10ca_1^2a_2^3+5ca_0a_2^4 \;=\;& 0,\end{aligned} \tag{39}$$

$$(3w^4-2w^2+3)[32ka_1a_2^2A^2+44kABa_2^3]+5ca_1a_2^4 \;=\; 0, \tag{40}$$

$$24ka_2^3A^2(3w^4-2w^2+3)+ca_2^5 \;=\; 0. \tag{41}$$

Solving Eq. (41), we get

$$a_2=\pm\sqrt{\frac{-24k\gamma}{c}}\;A \tag{42}$$

where for the mathematical convenience, we define

$$\gamma\equiv 3w^4-2w^2+3$$

Next, we solve Eq. (40) to obtain the value of a_1 as

$$a_1=\frac{1}{2}\frac{B}{A}a_2$$

for $B = 0$, we get $a_1 = 0$. From Eq. (39), we obtain

$$a_0 = \mp\sqrt{-\frac{8\gamma k}{3c}}\ C \tag{43}$$

From Eq. (38), we obtain $0 = 0$ and Eq. (37) yield

$$b = 32\gamma k \left(AD - \frac{1}{3}C^2 \right) \tag{44}$$

Equation (36) yield 0=0 and from Eq. (35), we obtain

$$24kCD\gamma a_2^2 + 12A\psi - 3ba_0a_2 - \frac{80}{3}\gamma kC^2 a_0 a_2 = 0 \tag{45}$$

Equation (34) yield 0=0 and from Eq. (33), we obtain the constraining relation among the parameters as

$$ac = 192\gamma^2 k^2 A^2 D^2 + \frac{448}{9} \times \gamma^2 k^2 C^4 - 256\gamma^2 k^2 C^2 AD - 8\psi Cc \tag{46}$$

From Eq. (32), we obtain 0=0 and Eq. (31) yield

$$4D\psi a_2 + aa_0 = a_0^3 \left(b + \frac{8\gamma k}{3}C^2 \right) \tag{47}$$

Let us now simplify Eq. (45) using Eqs. (42), (43) and (44), we obtain the constraining relation

$$\psi c = \gamma^2 k^2 C \left(\frac{32}{9}C^2 - 16\ AD \right) \tag{48}$$

Let us now take $B = D = 0$ in the set of Eqs. (31) to (41). For $B = D = 0$, Eqs. (44) and (47) yield

$$C = \left(\frac{3ac}{64\gamma^2 k^2} \right)^{\frac{1}{4}} = \frac{(3ac)^{\frac{1}{4}}}{\sqrt{8\gamma k}} \tag{49}$$

From Eq. (48), we obtain the constraining relation

$$\psi c = \frac{32}{9}\ \gamma^2 k^2 C^3$$

Let us discuss the following special cases of Eq. (7).

Case (2a): For the case when $B = D = 0$, the solution of Eq. (7) is given by [Ref]

$$z(\xi) = \sqrt{-\frac{C}{A}}\ \text{sech}\left(\sqrt{C}\ \xi \right)$$

Note that the variable x in Eq. (7) is now replaced by the variable ξ. Thus the solution $\phi(\xi)$ from Eq. (8) becomes

$$\begin{aligned}
\phi(\xi) &= a_0 + a_1 z(\xi) + a_2 z^2(\xi) = a_0 + a_2 z^2(\xi) \\
&= \mp\sqrt{\frac{-8\gamma k}{c}}\; C \pm \left(-\frac{C}{A} \times \sqrt{-\frac{24k\gamma}{c}}\; A \right) \operatorname{sech}^2\left(\sqrt{C}\;\xi\right) \\
&= \mp\sqrt{\frac{-8\gamma k}{c}} \times \frac{(3ac)^{\frac{1}{4}}}{\sqrt{8\gamma k}} \mp \sqrt{-\frac{24k\gamma}{c}} \times \frac{(3ac)^{\frac{1}{4}}}{\sqrt{8\gamma k}}\; \operatorname{sech}^2\left(\sqrt{C}\;\xi\right) \\
&= \mp(3ac)^{\frac{1}{4}} \times \sqrt{-\frac{1}{c}}\left[1 + \sqrt{3}\operatorname{sech}^2\left(\sqrt{C}\;\xi\right)\right] \\
&= \mp(3ac)^{\frac{1}{4}} \times \sqrt{-\frac{1}{c}}\left[1 + \sqrt{3}\operatorname{sech}^2\left\{\left(\frac{3ac}{64\gamma^2 k^2}\right)^{\frac{1}{8}}\;(x - wt)\right\}\right]
\end{aligned}$$

Note that this is the solitary wave solution of Eq. (30), provided that $a < 0$ and $c < 0$. From Eq. (44), one can see that, if $a < 0$ and $c < 0$, then b is also less than 0. On the other hand, if $c > 0$, then we get the imaginary solution, which is physically not acceptable. Also, note that the obtained solitary wave is not static (see Ref. [8] for further details).

Case (2b): Next we take $A = 1$, $B = 0$, $C = 2s$ and $D = s^2$, (where s is another constant) then Eq. (7) admits the solution as

$$z(\xi) = -\sqrt{-s}\;\tanh(\sqrt{-s}\;\xi)$$

For this particular choice of parameters, we obtain the value of b from Eq. (44) as

$$b = -\frac{32}{3}\;\gamma k s^2 \tag{50}$$

Using Eq. (48), we obtain the constraining relation as

$$\psi c = -\frac{32}{9}\;\gamma^2 k^2 s^3 \tag{51}$$

Simplifying Eq. (47), we obtain $a = -6s\psi$ and from Eqs. (46) and (51), we get

$$ac = \frac{192}{9}\;\gamma^2 k^2 s^4 = \frac{64}{3}\;\gamma^2 k^2 s^4$$

From here, we obtain the value of s as

$$s = \left(\frac{3ac}{64\gamma^2 k^2}\right)^{\frac{1}{4}} = \frac{(3ac)^{\frac{1}{4}}}{\sqrt{8\gamma k}}$$

Thus the solution (8) of Eq. (30) is given by

$$\begin{aligned}
\phi(\xi) &= a_0 + a_1 z(\xi) + a_2 z^2(\xi) = a_0 + a_2 z^2(\xi) \\
&= \mp\sqrt{-\frac{8\gamma k}{3c}} \times (2s) \pm \sqrt{-\frac{24\gamma k}{c}} \times \left[-s \ \tanh^2(\sqrt{-s}\ \xi)\right] \\
&= \mp\sqrt{-\frac{8\gamma k}{3c}} \times \left(2 \cdot \frac{(3ac)^{\frac{1}{4}}}{\sqrt{8\gamma k}}\right) \mp \sqrt{-\frac{24\gamma k}{c}} \times \frac{(3ac)^{\frac{1}{4}}}{\sqrt{8\gamma k}} \ \tanh^2(\sqrt{-s}\ \xi) \\
&= \mp(3ac)^{\frac{1}{4}} \times \sqrt{-\frac{1}{c}} \left[\sqrt{\frac{4}{3}} + \sqrt{3}\ \tanh^2\left\{\left(-\frac{3ac}{64\gamma^2 k^2}\right)^{\frac{1}{8}} (x - wt)\right\}\right]
\end{aligned}$$

Again this is the solitary wave solution of Eq. (30), provided that $a < 0$ and $c < 0$. From Eq. (50), one can see that b is also less than 0. If $c > 0$, then we get the imaginary solution. Again, the obtained solitary wave solution is not static [8].

Case (2c): Let us take $A = m^2,\ B = 0,\ C = -(1 + m^2)$ and $D = 1$ with $(0 < m^2 < 1)$ in Eq. (7), then the solution of (7) is given by [9] $z(\xi) = \mathrm{sn}(\xi)$. Substituting the value of $A,\ D$ and C^2 in Eq. (44), we obtain the value of b as

$$b = \frac{32\gamma k}{3}(m^2 - m^4 - 1)$$

Equation (48) yields the constraining relation among the parameters as

$$\psi c = \frac{16\gamma^2 k^2}{9}(3m^4 - 2m^6 + 3m^2 - 2)$$

Using Eq. (46), we obtain the value of ac as

$$\begin{aligned}
ac &= \frac{\gamma^2 k^2}{9}\left[192\ (m^8 + 1) - 384\ (m^6 + m^2) + 576\ m^4\right] \\
&= \frac{192\gamma^2 k^2}{9}\left[(m^8 + 1) - 2(m^6 + m^2) + 3\ m^4\right] \\
&= \frac{64\gamma^2 k^2}{3}\left[(m^8 + 1) - 2(m^6 + m^2) + 3\ m^4\right]
\end{aligned}$$

Finally, the solution (8) of Eq. (30) becomes

$$\begin{aligned}
\phi(\xi) &= a_0 + a_1 z(\xi) + a_2 z^2(\xi) = a_0 + a_2 z^2(\xi) \\
&= \pm\sqrt{-\frac{8k\gamma}{3c}}\ (1 + m^2) \pm \sqrt{-\frac{24k\gamma}{c}} \times m^2\ \mathrm{sn}^2(\xi) \\
&= \pm\sqrt{-\frac{8k\gamma}{3c}}\left[1 + m^2 + 3m^2\ \mathrm{sn}^2(\xi)\right]
\end{aligned}$$

This equation represents periodic wave solution of Eq. (30). In the limit $m \to 1$, $\mathrm{sn}(\xi) \to \tanh(\xi)$, and hence the solitary wave solution of Eq. (30) becomes

$$\phi(\xi) = \pm\sqrt{-\frac{8k\gamma}{3c}}\left[2 + 3\ \tanh^2(\xi)\right]$$

This solution is solitary wave for $c < 0$.

Case (2d): If we take $A = -m^2$, $B = 0$, $C = 2m^2 - 1$ and $D = 1 - m^2$ with $(0 < m^2 < 1)$ in Eq. (7), then the solution of (7) is given by [9] $z(\xi) = \mathrm{cn}(\xi)$. Using Eq. (44), we obtain the value of b as

$$b = \frac{32\gamma k}{3}(m^2 - m^4 - 1)$$

Solving Eq. (48), we obtain the value of ψc as

$$\psi c = \frac{16\gamma^2 k^2}{9}(3m^4 - 2m^6 + 3m^2 - 2)$$

Equation (46) yields the value of ac as

$$ac = \frac{64\gamma^2 k^2}{3}\left[(m^8 + 1) - 2(m^6 + m^2) + 3\ m^4\right]$$

and hence the solution (8) of Eq. (30) becomes

$$\begin{aligned}\phi(\xi) &= a_0 + a_1 z(\xi) + a_2 z^2(\xi) = a_0 + a_2 z^2(\xi)\\ &= \mp\sqrt{-\frac{8k\gamma}{3c}}\ (2m^2 - 1) \mp \sqrt{-\frac{24k\gamma}{c}} \times m^2\ \mathrm{cn}^2(\xi)\\ &= \mp\sqrt{-\frac{8k\gamma}{3c}}\left[2m^2 - 1 + 3m^2\ \mathrm{cn}^2(\xi)\right]\end{aligned}$$

which again represents periodic wave solution of Eq. (30). In the limit $m \to 1$, $\mathrm{cn}(\xi) \to \mathrm{sech}(\xi)$, and hence the solitary wave solution of Eq. (30) becomes

$$\phi(\xi) = \mp\sqrt{-\frac{8k\gamma}{3c}}\left[1 + 3\ \mathrm{sech}^2(\xi)\right]$$

The obtained solution is physically acceptable for $c < 0$.

Case (2e): If we take $A = -1$, $B = 0$, $C = 2 - m^2$ and $D = m^2 - 1$ with $(0 < m^2 < 1)$ in Eq. (7), then the solution of (7) is given by [9] $z(\xi) = \mathrm{dn}(\xi)$. In this case Eq. (44) becomes

$$b = \frac{32\gamma k}{3}(m^2 - m^4 - 1)$$

and Eq. (48) yields

$$\psi c = \frac{16\gamma^2 k^2}{9}(3m^4 - 2m^6 + 3m^2 - 2)$$

Equation (46) yields the value of ac as

$$ac = \frac{64\gamma^2 k^2}{3}[(m^8 + 1) - 2(m^6 + m^2) + 3m^4]$$

and hence the Eq. (8) yields the periodic wave solution of Eq. (30) as

$$\begin{aligned}
\phi(\xi) &= a_0 + a_1 z(\xi) + a_2 z^2(\xi) = a_0 + a_2 z^2(\xi) \\
&= \mp\sqrt{-\frac{8k\gamma}{3c}}\,(2 - m^2) \mp \sqrt{-\frac{24k\gamma}{c}}\,\mathrm{dn}^2(\xi) \\
&= \mp\sqrt{-\frac{8k\gamma}{3c}}\,\left[2 - m^2 + 3\,\mathrm{dn}^2(\xi)\right]
\end{aligned}$$

When $m \to 1$, $\mathrm{dn}(\xi) \to \mathrm{sech}(\xi)$, and the solitary wave solution of Eq. (30) becomes

$$\phi(\xi) = \mp\sqrt{-\frac{8k\gamma}{3c}}\,\left[1 + 3\,\mathrm{sech}^2(\xi)\right]$$

Again note that the solution is physically acceptable only when $c < 0$.

II. CONCLUDING REMARKS

The solitary wave solution of the non-canonical lagrangian for ϕ^6 potential have been obtained using the auxiliary equation method. The auxiliary equation method does not give a solution of non-canonical Lagrangian with ϕ^6 potential. To find the solution of this equation, we have to use some perturbative technique, like reductive perturbation technique [10], modified reductive perturbation technique [11], multiple-time scale perturbation method [12], homotopy perturbation method [13], etc. On the other hand, in the the presence of gradient term (or Ginzburg term) we obtained the solitary and periodic wave solutions of the non-canonical Lagrangian with ϕ^6 potential. Also note that the obtained solution is physically acceptable for $a < 0,\ b < 0$ and $c < 0$ (see cases (2a)-(2e)).

The canonical Lagrangian with Ginzburg term have been extensively studied in the context of field theory, cosmology and condensed matter physics. For the first time in this work, we have obtained the solitary and periodic wave solutions of the non-canonical Lagrangian with ϕ^6 potential and Ginzburg term. Since the non-canonical Lagrangian has been studied only in the context of cosmology, the result obtained here may help us to explain some new phenomena in field theory and condensed matter physics. Also, the exact solutions obtained here may provide a background for developing the perturbative solutions for several related equations. It would also be interesting to find the solution of non-canonical Lagrangian with coupled ϕ^6 potential [14,15,16]. Such studies are in progress.

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