UNIT 1

LINEAR PROGRAMMING

OUTLINE

OBJECTIVES:

By the end of the unit**,** you should be able to:

1. Identify and formulate Linear Programming Problems
2. Solve Linear Programming Problems using the graphical method
3. Conduct and explain sensitivity analysis
4. Formulate and solve the dual problem
5. Explain dual (shadow) prices

Note: In order to achieve these objectives, you need to spend a minimum of four (4) hours and a maximum of six (6) hours working through the sessions.

SESSION 5.1: INTRODUCTION

Linear programming was developed by applied mathematicians and operations research specialists as a means to solve real-world problems using linear methods. Based on the fundamentals of matrix algebra, linear programming seeks to find an optimal solution, using quantitative methods, to a particular problem given a finite number of constraints. It is used extensively in managerial science and has widespread utility to business, government, and industry. It is applied especially to problems in which decision-makers wish to minimize costs or maximize profits under a given operating construct. In many cases, linear programming will affect decisions regarding materials used in manufacturing and construction or even the hiring of personnel or particular skill sets. It is an excellent tool for decisions regarding allocation of scarce resource.

In mathematics, L**inear Programming** (LP) problems involve the optimization of a linear objective function, subject to linear equality and inequality constraints. Put very informally, LP is about trying to get the best outcome (e.g. maximum profit, least effort, etc) given some list of constraints (e.g. only working 30 hours a week, not doing anything illegal, etc), using a linear mathematical model.

Linear programming is based on linear equations—equations of variables raised only to the first power. Many variables--such as x, y, and z) or more generically, x_1, x_2,..., x_n may be used in a single equation, known as a **linear combination**. Generally, each **variable** represents a quantifiable item, such as the number of carpenters' hours available to a business during a week or number of sleds available for sale during a given month.

Many operations management decisions involve trying to make the most effective use of an organisation's resources. Resources typically include machinery (such as planes in the case of an airline), labour (such as pilots), money, time, and raw materials (such as jet fuel). These resources may be used to produce products (such as machines, furniture, food, and clothing) or service (such as airline schedules, advertising policies, or investment decisions).

SESSION 5.2: WHAT IS LINEAR PROGRAMMING

Linear Programming, a specific class of mathematical problems, in which a linear function is maximized (or minimized) subject to given linear constraints. This problem class is broad enough to encompass many interesting and important applications, yet specific enough to be tractable even if the number of variables is large.

Linear Programming (or simply LP) refers to several related mathematical techniques that are used to allocate limited resources among competing demands in an optimal way.

The word linear means that all the mathematical functions in this model are required to be linear functions. The term programming here does not imply computer programming, rather it implies planning. Thus, Linear Programming (LP) means planning with a linear model. It refers to several related mathematical techniques that are used to allocate limited resources among competing demands in an optimal way.

The objective of LP is to determine the optimal allocation of scarce resources among competing products or activities in a best possible(i.e. optimal) way. That is, it is concerned with the problem of optimizing (minimizing or maximizing) a linear function subject to a set of constraints in the form of inequalities. Economic activities call for optimizing a function subject to several inequality constraints.

SESSION 1.3: APPLICATIONS OF LINEAR PROGRAMMING

1. **Product Planning:** Finding the optimal product mix where several products have different costs and resource requirements (for example, finding the optimal blend of constituents for gasoline, paints, human diets, animal feed).

6. **Product Routing:** Finding the optimal routing for a product that must be processed sequentially through several machine centres, with each machine in a centre having its own cost and output characteristics.
7. **Process Control:** Minimizing the amount of scrap material generated by cutting steel, leather, or fabric from a roll or sheet of stock material.

8. **Inventory Control:** Finding the optimal combination of products to stock in a warehouse or store.
9. **Distribution Scheduling:** Finding the optimal shipping schedule for distributing products between factories and warehouses or warehouses and retailers.
10. **Plant Location Studies:** Finding the optimal location of a new plant by evaluating shipping costs between alternative locations and supply and demand sources.
11. **Materials Handling:** Finding the minimum-cost routings of material handling devices(such as forklift trucks) between departments in a plant and of hauling materials from a supply yard to work site by trucks, with each truck having different capacity and performance capabilities.

SESSION 1.4: EXAMPLES OF LP PROBLEMS

1. A Product Mix Problem

- A manufacturer has fixed amounts of different resources such as raw material, labor, and equipment.
- These resources can be combined to produce any one of several different products.
- The quantity of the *ith* resource required to produce one unit of the *jth* product is known.
- The decision maker wishes to produce the combination of products that will maximize total income.

2. A Blending Problem

- Blending problems refer to situations in which a number of components (or commodities) are mixed together to yield one or more products.
- Typically, different commodities are to be purchased. Each commodity has known characteristics and costs.

- The problem is to determine how much of each commodity should be purchased and blended with the rest so that the characteristics of the mixture lie within specified bounds and the total cost is minimized.

3. A Production Scheduling Problem

- A manufacturer knows that he must supply a given number of items of a certain product each month for the next *n* months.
- They can be produced either in regular time, subject to a maximum each month, or in overtime. The cost of producing an item during overtime is greater than during regular time. A storage cost is associated with each item not sold at the end of the month.
- The problem is to determine the production schedule that minimizes the sum of production and storage costs.

4. A Transportation Problem

- A product is to be shipped in the amounts *al, a2, ..., am* from *m* shipping origins and received in amounts *bl, b2, ..., bn* at each of *n* shipping destinations.
- The cost of shipping a unit from the ith origin to the jth destination is known for all combinations of origins and destinations.
- The problem is to determine the amount to be shipped from each origin to each destination such that the total cost of transportation is a minimum.

5. A Flow Capacity Problem

- One or more commodities (e.g., traffic, water, information, cash, etc.) are flowing from one point to another through a network whose branches have various constraints and flow capacities.
- The direction of flow in each branch and the capacity of each branch are known.
- The problem is to determine the maximum flow, or capacity of the network.

SESSION 1.5: REQUIREMENTS OF A LINEAR PROGRAMMING PROBLEM

All LP problems have four properties in common:

i. LP problems seek to maximize or minimize some quantity (usually profit and cost). We refer to this property as the **objective function** of an LP problem. The major objective of a typical firm is to minimize dollar profits in the long run. In the case of a trucking or airline distribution system, the objective might be to minimize shipping cost.

ii. The presence of a **restriction, or constraint**, limits the degree to which we can pursue our objective. For example, deciding how many units of each product in a firm's product line to manufacture is restricted by available labour and machinery. We want, therefore to maximize or minimize a quantity (the objective function) subject to limited resources (the constraints)

iii. There must be **alternative courses of action** to choose from. For example, if a company produces three different products, management may use LP to decide how to allocate among them its limited production resources (of labour, machinery, and so on). If there were no alternatives to select from, we would not need LP.

iv. The objective and constraints in Linear Programming problems must be expressed in terms of **linear equations or inequalities.**

SESSION 1.6: ASSUMPTIONS OF LINEAR PROGRAMMING

1. **Proportionality:** The contribution of each activity to the value of the objective function Z is proportional to the level of the activity
2. **Additivity:** Every function in a linear programming model is the sum of the individual contributions of the respective activities.
3. **Divisibility:** Decision variables in a linear programming model are allowed to have any values including no integer values that satisfy the functional and nonnegative constraints. These values are not restricted to just integer values. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels.
4. **Certainty:** The value assigned to each parameter of a linear programming model is assumed to be a known constant

SESSION 1.7: TERMINOLOGIES

1. **Decision Variables**: The unknown of the problem whose values are to be determined by the solution of the LP. In mathematical statements we give the variables such names as $X_{1,}\ X_{2,}\ X_{3,} \dots X_n$
2. **Objective Function:** The measure by which alternative solutions are compared. The general objective function can be written as :

$$Z = C_1 X_1 + C_2 X_2 + C_3 X_3 + \dots + C_n X_n$$

The measure selected can be either maximized or minimized.

The first step in LP is to decide what result is required. This may be to minimize cost / time, or to maximize profit/contribution. Having decided upon the objective, it is now necessary to state mathematically the elements involved in achieving this. This is called the objective function as noted above.

EXAMPLE: A factory can produce two products, A and B. The contribution that can be obtained from these products are: "A contributes \$20 per unit and B contributes \$30 per unit" and it is required to maximize contribution.

Let the decisions be X_1 and X_2. Then the objective function for the factory can be expressed as:

$$Z = 20X_1+30X_2$$

where

X_1 = number of units of A produced.

X_2 = number of units of B produced

This problem has two (2) unknowns. These are called decision variables.

Note that only a single objective (in the above example, to maximize contributions) can be dealt with at a time with an LP problem.

3. **Constraint**: A linear inequality defining the limitations on the decisions. Circumstances always exist which govern the achievement of the objectives. These factors are known as limitations or constraints. The limitation in any problem must

clearly be identified, quantified and expressed mathematically. To be able to use LP, they must, of course, be linear.

4. Non-negative restriction: Solution algorithms assume that the variables are constrained to be non-negative ie $X_j \geq 0$, for j = 1, 2, 3, .., n.

5. Optimal solution: A feasible solution that maximizes/minimizes the objective function. It is the solution that has the most favourable value of the objective function.

5. Alternative optimal solution: If there are more than one optimal solution (with the same value of Z), the model is said to have alternative optimal solution.

7. Feasible solutions: The set of points (solutions) satisfying the LP's constraints.

SESSION 1.8: STANDARD FORM OF THE MODEL

The standard form adopted is:

For maximization problems we have:

Maximize: $Z = 30X_1 + 40X_2 + 20X_3 \longrightarrow$ Objective function

$$\text{Subject to: } \left.\begin{array}{l} 2X_1 + 2X_2 - X_3 \leq 16 \\ 4X_1 + 5X_2 - X_3 \leq 10 \\ 7X_1 + 3X_2 - X_3 \leq 30 \end{array}\right\} \longrightarrow \text{Constraints}$$

$$X_1, X_2, X_3 \geq 0 \longrightarrow \text{Non-negative restriction}$$

For maximization problems we have:

Minimize: $Z = 30X_1 + 40X_2 + 20X_3 \longrightarrow$ Objective function

$$\text{Subject to: } \left.\begin{array}{l} 3X_1 + 2X_2 - X_3 \geq 6 \\ 4X_1 + 5X_2 - X_3 \geq 6 \\ 6X_1 + 2X_2 - X_3 \geq 3 \end{array}\right\} \longrightarrow \text{Constraints}$$

$$X_1, X_2, X_3 \geq 0 \longrightarrow \text{Non-negative restriction}$$

SESSION 1.9: FORMULATING LINEAR PROGRAMMING PROBLEMS

One of the most common linear programming applications is the product – mix problem. Two or more products are usually produced using limited resources. The company would like to determine how many units of each product it should produce in order to maximize overall profit given its limited resources. Let us look at an example:

Procedure:

Formulating Linear Programming problems means selecting out the important elements from the problem and defining how these are related. For real -world problems, this is not an easy task. However, there are some steps that have been found useful in formulating Linear Programming problems:

i. Identify and define the unknown variables in the problem. These are the decision variables.
ii. Summarize all the information needed in the problem in a table
iii. Define the objective that you want to achieve in solving the problem. For example, it might be to reduce cost (minimization) or increase contribution to profit (maximization). Select only one objective and state it.
iv. State the constraint inequalities.

Example:

A manufacturing company produces two products- wax and yarn. Each wax takes 4 hours in the dying department, and 2 hours in the packaging department. Each yarn requires 3 hours in dying department and1 hour in packaging department. During the current production period, 240 hours of production time are available in the dying department and 100 hours of time production time are also available in the packaging department.

Each wax produced and sold yields a profit $7and each yarn produced may be sold for a profit of $5

SOLUTION

Step 1: **Identify and define the unknown variables (decision variables) in the problem**

Let the decision variables be X_1 and X_2. Then

X_1= number of wax to be produced

X_2 = number of yarn to be produced

Step 2: Summarize the information needed in a table.

	Hours required to produce 1 unit		
Department	Wax (X_1)	Yarn ($X_{2)}$	Hours available
Electronic	4	3	240
Assemble	2	1	100
Profit per unit	\$7	\$5	

Step3: Define the objective that you want to achieve in solving the problem. State the LP objective function in terms of X_1 and X_2 as follows:

Maximize profit, $P = \$7X_1 + \$5X_2$

Step 4: State the constraint inequalities

Our next step is to develop mathematical relationships to describe the two constraints in this problem. One general relationship is that the amount of a resource used is to be less than or equal to (≤) the amount of resource available.

First constraint: Dying time used is ≤ dying time available.

$4X_1 + 3X_2 \leq 240$(hours of dying time available)

Second constraint: Packaging time used is ≤ Packaging time available

$2X_1 + 1X_2 \leq 100$(hours of packaging time available)

The above LP problem is stated as follows:

Maximize profit, $P = \$7X_1 + \$5X_2$

Subject to the constraints:

$4X_1 + 3X_2 \leq 240$(hours of dying time available)

$2X_1 + 1X_2 \leq 100$(hours of packaging time available

$X_1, X_2 \geq 0$

SESSION 1.10: SOLVING LINEAR PROGRAMMING: GRAPHICAL METHOD

For optimization subject to a single inequality constraint, the Lagrangian method is relatively simple. When more than one inequality constraints are involved, Linear Programming is easier. If the constraints, however numerous, are limited to two variables, the easiest solution is the graphical method. If the variables are more than two, then an algebraic method known as the simplex method is used.

Example 1

Maximize profit, P = \$7x + \$5yy

Subject to the constraints:

$2x + y \leq 32$ (hours of electronic time available)

$2x + y \leq 18$ (hours of assembly time available

$x, y \geq 0$

Solution: Treat the inequality constraints as equations and find the intersections of each on the axes. Proceed as follows:

<u>$2x + y \leq 32$</u> (hours of electronic time available)

1. On the On the x-axis, $y = 0$

$\Rightarrow 2x + 0 = 32$

$\therefore x = 16$

So coordinate on the y-axis is (16, 0)

2. On the y , $x = 0$

$\Rightarrow 0 + y = 32$

$\therefore y = 32$

So coordinate on the y-axis is (0, 32)

<u>$x + y \leq 18$</u> (hours of assembly time available)

1. On the On the x-axis, $y = 0$

$\Rightarrow x + 0 = 18$

$\therefore x = 18$

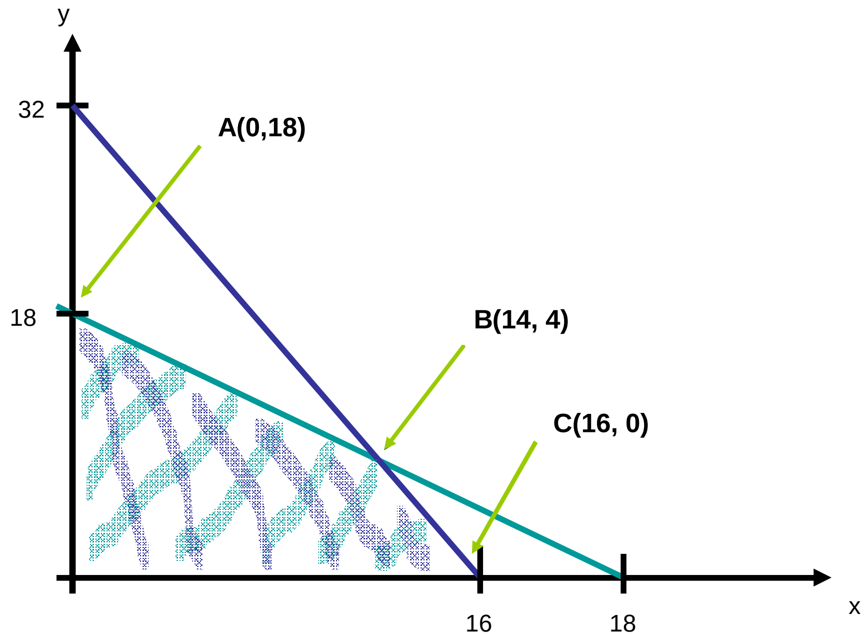
So coordinate on the y-axis is (18, 0)

2. On the y , x = 0

$\Rightarrow 0 + y = 18$

$\therefore$ y= 18

So coordinate on the y-axis is (0, 18)



NOTE

The shaded area is called the feasible region. It contains all the points that satisfy all two constraints plus the non-negative constraints. Z is maximized at the intersection of the two constraints called an extreme point. The co – ordinate that maximizes the objective function is called feasible solution.

The mathematical theory behind Linear Programming states that an optimal solution to any problem (that is the value that yields the maximum profit) will lie at the corner point or extreme point of the feasible region. Hence it is necessary to find only the values at each corner point.

From the graph, the corner points are: **A(0, 18), B(14, 4), and C(16, 0). Substituting the co-ordinates of the corner points into the objective function, z = 80x + 70y,** we have the following:

A(0, 18): Z=80(0) + 70(18) = 1260

B(14, 4): Z=80(14) + 70(4) = 1400

C(16, 0): Z=80(16) + 70(0) = 1280

- In order to maximize profit:
  - 14 units of X
  - 4 units of y must be produced.

Example 2

Maximize $z = 80x + 70y$

Subject to the constraints:

$2x + y \leq 32$(Constraint A)

$x + y \leq 18$(Constraint B)

$4x + 12y \leq 144$(Constraint C)

$x, y \leq 0$

Solution: Treat the inequality constraints as equations and find the intersections of each on the axes. Proceed as follows:

2x + y =32(Constraint A)

1. On the On the x-axis, y = 0

$\Rightarrow 2x + 0 = 32$

$\therefore x = 16$

So coordinate on the y-axis is (16, 0)

2. On the y , x = 0

$\Rightarrow 0 + y = 32$

$\therefore y = 32$

So coordinate on the y-axis is (0, 32)

x + y = 18(Constraint B)

1. On the On the x-axis, y = 0

$\Rightarrow x + 0 = 18$

$\therefore x = 18$

So coordinate on the y-axis is (18, 0)

2. On the y , x = 0

$\Rightarrow 0 \ + \ y = \ 18$

$\therefore$ y= 18

So coordinate on the y-axis is (0, 18)

<u>4x + 12y = 144</u>(Constraint C)

On the On the x-axis, y = 0

$\Rightarrow 4x \ + \ 0 \ = \ 144$

$\therefore$ x= 36

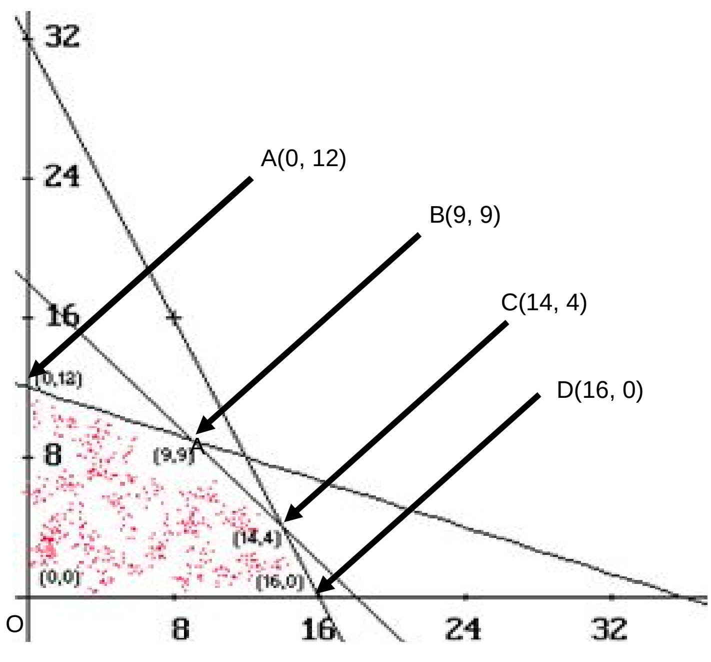
So coordinate on the y-axis is (36, 0)

2. On the y , x = 0

$\Rightarrow 0 \ + \ 12y = \ 144$

$\therefore$ y= 12

So coordinate on the y-axis is (0, 12)



From the graph, the corner points are: **A(0, 12), B(9, 9), C(14, 4) and D(16, 0).**

Substituting the co-ordinates of the corner points into the objective function,

z = 80x + 70y, we have the following:

Z=80(0)+70(12)=840

Z=80(9)+70(9)=1350

Z=80(14)+70(4)=1400

Z=80(16)+70(0)=1280

Therefore maximum profit is $1400, when 14 of y and 4 of x are

Example 3

A fabric firm has received an order for cloth specified to contain at least 45 pounds of cotton and 25 pounds of silk. The cloth can be woven out of any suitable mix of two yarns (A and B). Material A costs $3 per pound, and B costs $2 per pound. They contain the proportions of cotton and silk (by weight) as shown in the following table:

	Cotton	Silk
A	30%	50%
B	60%	10%

Required:

i. Formulate the linear programming problem.
ii. Find the quantities (pounds) of A and B that should be used to minimize the cost of this order.

Solution

Let x = pounds of material A produced

Let y = pounds of material B produced

Objective function: Min C = 3x + 2y

Constraints: $.30x + .60y \geq 45$

$.50x + .10y \geq 25$

Finding the intersections of the constraints on the axes

$.30A + .60B \geq 45$

On the On the x-axis, $y = 0$

$\Rightarrow ,30x + 0 = 45$

$\therefore$ x= 150

So coordinate on the y-axis is (150, 0)

2. On the y , $x = 0$

$\Rightarrow 0 + .60y = 45$

$\therefore$ y= 75

So coordinate on the y-axis is (0, 75)

$.50x + .10y \geq 25$

On the On the x-axis, $y = 0$

$\Rightarrow ,50x + 0 = 25$

$\therefore$ x= 50

So coordinate on the y-axis is (50, 0)

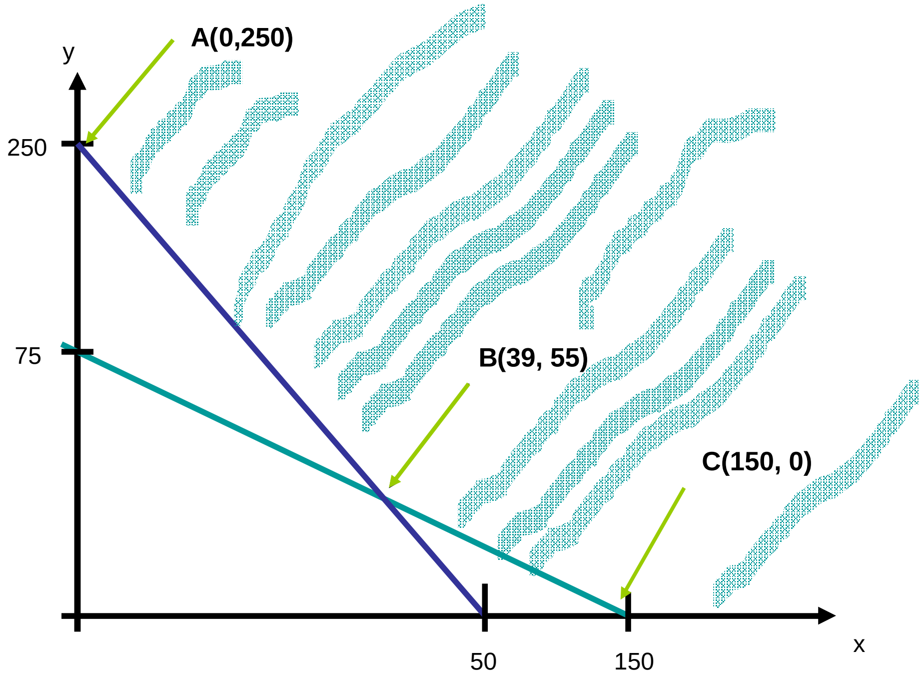
2. On the y , $x = 0$

$\Rightarrow 0 + .10y = 25$

$\therefore$ y= 250

So coordinate on the y-axis is (0, 250)

Graph the intersections of the constraints



From the graph, the corner points are: **A (0, 250), B (39, 59), and C (150, 0):**

Substituting the co-ordinates of the corner points into the objective function,

C = 3x + 2y, we have the following:

A (0, 250): c=3(0) + 2(250) = 500

B (39, 59): c=3(39) + 2(55) = 227

C (150, 0): c=3(150) + 2(0) = 450

- In order to minimize cost:
  - 39 pounds of A
  - 55 pounds of B must be used.

<u>Example 3</u>

Maximize $Z = 10x_1 + 15x_2$

Subject to the constraints:

$3x_1 + 4x_2 = 10 ......Constraint\ A$

$x_1 + 4x_2 = 4 ....Constraint\ B$

<u>Constraint A</u>: $3x_1 + 4x_2 = 10$

Finding the intercept on the x_1-axis.

On the x_1-axis, $x_2 = 0$

$3x_1 + 4(0) = 10$

$3x_1 = 10$

$\therefore x_1 = \frac{10}{3} = 3.3$ (Co-ordinate: 3.3, 0)

Finding the intercept on the x_2-axis.

On the x_2-axis, $x_1 = 0$

$3(0) + 4x_2 = 10$

$4x_2 = 10$

$\therefore x_2 = \frac{10}{4} = 2.5$ (Co-ordinate: 0, 2.5)

<u>Constraint B</u>: $x_1 + 4x_2 = 4$

Finding the intercept on the x_1-axis.

On the x_1-axis, $x_2 = 0$

$x_1 + 4(0) = 4$

$\therefore x_1 = 4$ (Co-ordinate: 4, 0)

Finding the intercept on the x_2-axis.

On the x_2-axis, $x_1 = 0$

$0 + 4x_2 = 4$

$4x_2 = 4$

$\therefore x_2 = \frac{4}{4} = 1$ (Co-ordinate: 0, 1)

NOTE: Draw the graph to check the co-ordinates

The co-ordinates of the feasible region are: P(0, 1),Q (3, 0.25) and R(3.33, 0)

Substitute these into the objective function $Z = 10x_1 + 15x_2$

P(0, 1): $Z = 10x_1 + 15x_2 = 10(0) + 15(1) = 15$

Q(3, 0.25): $Z = 10x_1 + 15x_2 = 10(3) + 15(0.25) = 30 + 3.75 = 33.75$

R(3.33, 0): $Z = 10x_1 + 15x_2 = 10(3.33) + 15(0) = 33.33$

The point Q(3, 0,25) yields the maximum Z. So, in order to maximize profit, 3 units of x_1 and 0.25 units of x_2 must be produced. This will yield a profit of $33.75

<u>Example 4</u>

Maximize

$Z = 3X_1 + 4X_2$

Subject to:

$2.5X_1 + X_2 \leq 20$(Constraint A)

$3X_1 + 3X_2 \leq 30$(Constraint B)

$2X_1 + X_2 \leq 16$(Constraint C)

$X_1, X_2, \geq 0$

<u>Procedure:</u>

Treat the inequality constraints as equations and find the intersections of each on the axes.

<u>Constraint A :$2.5X_1 + X_2 = 20$</u>

1. On the X_1-axis, $X_2 = 0$

$\Rightarrow 2.5X_1 + 0 = 20$

$\therefore X_1 = 8$

So coordinate on the X_1-axis is (8, 0)

2. On the $\mathbf{X_2}$, $X_1 = 0$

$\Rightarrow 0 + X_2 = 20$

$\therefore X_2 = 20$

So coordinate on the X_2-axis is (0, 20)

Constraint B: $3X_1 + 3X_2 = 30$

1. On the X_1-axis, $X_2 = 0$

$\Rightarrow 3X_1 + 0 = 30$

$\therefore X_1 = 10$

So coordinate on the X_1-axis is (10, 0)

2. On the X_2 , $X_1 = 0$

$\Rightarrow 3X_1 + 0 = 30$

$\therefore X_2 = 10$

So coordinate on the X_1-axis is (0, 10)

Constraint C: $2X_1 + X_2 \leq 16$

1. On the On the X_1-axis, $X_2 = 0$

$\Rightarrow 2X_1 + 0 = 16$

$\therefore X_1 = 8$

So coordinate on the X_1-axis is (8, 0)

2. On the X_2 -axis, $X_1 = 0$

$\Rightarrow 0 + X_2 = 16$

$\therefore X_2 = 16$

So coordinate on the X_1-axis is (0, 16)

Graph the equations using their intersections in order to check the co-ordinates

EXAMPLE 5:

Maximize $8x_1 + 10x_2$

Subject to: $3x_1 + 5x_2 \leq 500$

$4x_1 + 2x_2 \leq 350$

$6x_1 + 8x_1 \leq 800$

$x_1 , x_2, \geq 0$

Finding the intercepts on the axes.

$3x_1 + 5x_2 = 500$

On the x_1-axis, $x_2=0$.

$\Rightarrow 3x_1 = 500$

$\therefore x = 166.7 = 167$

So coordinate on the x_1-axis is (167, 0)

On the $x_{2\,-}$ axis, x_1 =o

$\Rightarrow 5x_x = 500$

$\therefore\ x_2 = 100$

So coordinate on the x_2-axis is (0, 100)

$4x_1+2x_2= 350$

On the x_1 – axis, $x_2 = 0$.

$4x = 350$

$\therefore\ x_1 = 87.5$

So coordinate on the x_1-axis is(88, 0)

On the y – axis, x = 0

$\Rightarrow 2y= 350$

$\therefore\ y = 175$

So coordinate on the x_2-axis is(0, 175)

$6x+8y=800$

On the x –axis, y =0

$\Rightarrow 6x = 800$

$\therefore\ x = 133.3$ (133, 0

On the y – axis, x = 0

$\Rightarrow 8y = 800$

$\therefore\ y = 100$ (0, 100)

NOTE: Draw the graph to check the co-ordinates

Reading from the graph. The company should produce 60 units of x and 55 units of y in order to make a maximum profit of $1030

SESSION 1.11: SENSITIVITY ANALYSIS

Operations managers are usually interested in more than the optimal solution to an LP problem. In addition to knowing the value of each decision variable(the X_i's) and the value of the objective function, they want to know how sensitive these solutions are to input parameter(numerical value that is given in a model) changes. For example, what happens if the right-hand –side values of the constraints change?

Sensitivity analysis, or post optimality analysis, is the study of how sensitive solutions are to parameter(decision variable) changes.

It is an analysis that projects how much a solution might change if there were changes in the variables or input data.

Illustration

Recall example 3

Maximize $Z = 10x_1 + 15x_2$

Subject to the constraints:

$3x_1 + 4x_2 = 10 ......Constraint\ A$

$x_1 + 4x_2 = 4 ....Constraint\ B$

The optimum solution is: 3 units of x_1 and 0.25 units of x_2 and a profit of $33.75

The objective now is to find how much the profit will increase/decrease if the right-hand –side values of the constraints change.

Constraint A

Suppose the right-hand –side value of the constraint A changes from 10 to 11, how will the profit and the decision variables change?

The new LP problem is

Maximize $Z = 10x_1 + 15x_2$

Subject to the constraints:

$3x_1 + 4x_2 = 11......Constra\text{int } A$

$x_1 + 4x_2 = 5.........Constra\text{int } B$

Solving simultaneously, we have the following (please check): $x_1 = 3.5$ and $x_2 = 0.125$

Substitute the values into the objective function

$Z = 10x_1 + 15x_2 = 10(3.5) + 15(0.125) = 35 + 1.875 = 36.875 = 36.88$

<u>Interpretation</u>

If constraint A is increased by one more unit, 3.5 units of x_1 and 0.125 units of x_2 would be produced in order to make a profit of $36.88.

What will management do? Management will increase Constraint A because this will increase profit from $33.75 to $36.88. However, management must increase the production of x_1 from 3 to 3.5 units and increase that of x_2 from 0.25 to 0.125 units. Note that the difference in profit ($36.88- $33.75 = $3.13) is the dual price or shadow cost of constraint A. Compare these results with those under shadow prices.

<u>Constraint B</u>

Suppose the right-hand –side value of the constraint B changes from 4 to 5, how will the profit and the decision variables change?

The new LP problem is

Maximize $Z = 10x_1 + 15x_2$

Subject to the constraints:

$3x_1 + 4x_2 = 10......Constra\text{int } A$

$x_1 + 4x_2 = 5.........Constra\text{int } B$

Solving simultaneously, we have the following (please check): $x_1 = 2.5$ and $x_2 = 0.625$

Substitute the values into the objective function

$Z = 10x_1 + 15x_2 = 10(2.5) + 15(0.625) = 25 + 9.375 = 34.375 = 34.38$

Interpretation

If constraint B is increased by one more unit, 2.5 units of x_1 and 0.625 units of x_2 would be produced in order to make a profit of $34.38.

What will management do? Management will increase Constraint B because this will increase profit from $33.75 to $34.38. However, management must decrease the production of x_1 from 3 to 2.5 units and increase that of x_2 from 0.25 to 0.625 units. Note that the difference in profit ($34.38 - $33.75= $0.63) is the dual price or shadow cost of constraint B. Compare these results with those under shadow prices.

The same analysis can be done by increasing/decreasing any of the constraints.

SESSION 1.12: DUAL(SHADOW) PRICES

The shadow price, also called the dual, is the value of one (1) additional unit of a resource in the form of one (1) more hour of machine time, labour time, or other scarce resource. It answers the question: "Exactly how much should a firm be willing to pay to make additional resources available? Is it worthwhile to pay workers an overtime rate to stay one (1) extra hour each night in order to increase production output?

In order to answer this question, we need to formulate the dual problem. Every maximization (minimization) problem in Linear Programming has a corresponding minimization (maximization) problem. The original problem is called the primal and the corresponding problem is called the dual.

The following are the rules for transforming the primal to obtain the dual:

i. Reverse the inequality sign. That is maximization ($\leq$) in the primal becomes minimization ($\geq$) in the dual and vice versa. The non-negativity constraints on the decision variables is always maintained.
ii. The rows of the coefficient matrix of the constraints in the primal are transferred to columns for the coefficient matrix of the constraints in the dual.

iii. The row vector of coefficients in the objective function in the primal is transposed to a column vector of constraints for the dual constraints.

iv. The column vector of constraints from the primal constraints is transposed to a row vector of coefficients for the objective function of the dual.

Example 1

Maximize $Z = 10x_1 + 15x_2$

Subject to the constraints:

$3x_1 + 4x_2 \leq 10......Constra\,\text{int}\ A$

$x_1 + 4x_2 \leq 5.........Constra\,\text{int}\ B$

The dual of the above LP problem is:

Minimize $Q = 10A + 5B$

Subject to the constraints:

$3A + B \geq 10$ ……………(1)

$4A + 4B \geq 15$ …………..(2)

Solving simultaneously (assume the inequality to be equal)

Multiply (1) by 4

12A + 4B = 40……………..(3)

4A + 4B = 15…………….....(2)

Equation (3) –Equation (2)

8A = 25

$\therefore A = \frac{25}{8} = 3.126 = 3.13$

Put A = 3.125 into Equation (1)

3(3.125) + B = 10

9.375 + B = 10

$\therefore B = 10 - 9.375 = 0.625 = 0.63$

Interpretation of the results

The dual price of factor A is \$3.13. This is the value of one (1) additional unit of the resource. In other words, one (1) additional unit of resource A will yield a profit of \$3.13. The dual price of factor B is \$0.63. This is the value of one (1) additional unit of the resource. In other words, one (1) additional unit of resource B will yield a profit of \$0.63.

Example 2

We saw that the optimal solution example 2 is X_1 = 30 wax, X_2 = 40 yarn, and profit = \$410. Suppose the manager of the company is considering adding an extra person to work at the dying department at a salary of \$5.00 per hour. Should the firm do so?

Recall the LP problem from page 39 and 40.

Maximize $Z= 7X_1 + 5X_2$

Subject to: $4X_1 + 3X_2 \leq 240$(hours of dying time)

$2X_1 + 1X_2 \leq 100$(hours of packaging time)

The above LP problem is the primal. The dual is as follows:

Minimize: $C = 240d + 100p$

Subject to: $4d + 2p \leq 7$

$3d + 1p \leq 5$

Note: d = hours of dying time constraint

p = hours of packaging time constraint

Solving the dual equations simultaneously, we have the following: d = 0.5, and p = 1.5(please check)

Interpretation: If an extra worker in the dying department receives a salary of \$5.00 per hour, the company will lose \$4.50 for every hour the new worker works in the dying department. So the firm will not be willing to employ additional worker at a salary of \$5.00.

SELF ASSESSMENT QUESTIONS

QUESTION 1

Some students make necklaces and bracelets in their spare time and sell all they make. Every week, they have available 10,000 grams of metal and 20 hours to work. It takes 50 grams of metal to make a necklace and 200 grams to make a bracelet. Each necklace takes 30 minutes to make and each bracelet takes 20 minutes to make. The profit on each necklace is \$3.50 and the profit on each bracelet is \$2.50. The students want to make as much profit as possible.

Because you are taking a course in Operations Research, the students asked you for advice on the following:

i. What number of necklaces and bracelets should be made each week?

ii. How much profit can they make?

QUESTION 2

Jubilant manufacturing company Ltd. produces two types of decorative shelves. The Local English style takes 20 minutes to assemble and 10 minutes to finish. The Old Contemporary style takes 10 minutes to assemble and 20 minutes to finish. Each day, there are at least 48 worker – hours of labour available in the assembly department and at least 64 worker- hours of labour available in the finishing department

The cost of materials for the Local English shelf is \$2.00 each. The cost of material for the Old Contemporary shelf is \$2.50 each.

i. Formulate a Linear Programming problem for the above statement that will minimize cost of producing the two types of shelves.

ii. How many of each type of shelf should the company produce to minimize the cost of materials and still meet it's production commitments?

iii. Find the dual prices and interpret your results.

iv. Conduct a sensitivity analysis to find out how the optimum solution will respond to changes in the right-hand –side values of the constraints.

QUESTION 3.

A firm produces two products, X and Y with a contribution of $8 and $10 per unit respectively. Production data are shown in the table below:

	Labour Hours	Material A	Material b
X	3	4	6
Y	5	2	8
Total available	500	350	800

i. Formulate the LP for the statement above

ii. Solve the problem using the graphical method

iii. Calculate the shadow prices for the binding constraints and interpret your results.

iv. Conduct a sensitivity analysis to find out how the optimum solution will respond to changes in the right-hand –side values of the constraints.

QUESTION 4.

A manufacturer produces two products, Blocks and Bricks. Blocks have a contribution of $3 per unit and Bricks $4 per unit. The manufacturer wishes to establish the weekly production plan, which maximizes contribution. Production data are as follows:

	Machining (hours)	Labour (hours)	Material (kgs)
Klunck	4	4	1
Klinck	2	6	1
Total available	100	180	40

i. Formulate an LP model for the above problem.

ii. Solve the problem using the graphical method

iii. Find the dual prices and interpret your results. Conduct a sensitivity analysis to find out how the optimum solution will respond to changes in the right-hand –side values of the constraints.

QUESTION 5.

In a machine shop, a company manufactures two types of electronic components 'xlem' and yhart', on which it aims to maximize the contributions to profit. The company wishes to know the ideal combination of 'xlem' and 'yhart to make. All the electronic components are produced in three main stages: assembly, inspection and testing, and packing.

In the assembly each 'xlem' takes one hour and each 'yhart' takes two hours. Inspection and testing takes 7.5 minutes for each 'xlem' and 30 minute for each 'yhart' on average, which includes the time required for any faults to be rectified. There are 600 hours available for assembly and 100 hours for inspection and testing each week. At all stages both components can be processed at the same time.

At the final stage the components require careful packing prior to delivery. Each 'xlem' takes 3 minutes and each 'yhart' takes 20 minutes to pack properly. There is a total of 60 packing hours available each week. The contribution on 'xlem' is $10 per unit and on 'yhart' it is $15 per unit. For engineering reasons not more than 500 of 'xlem' can be produced. All production can be sold.

i. State the objective function in mathematical terms.
ii. State the constraint inequalities.
iii. Graph these constraints on a suitable graph and identify the feasible region.
iv. Advise the company on the optimal product mix and contribution.
v. Find the dual prices.
vi. Conduct a sensitivity analysis to find out how the optimum solution will respond to changes in the right-hand –side values of the constraints.

QUESTION 6.

The Shader Electronics Company produces two products: The Shader Walkman, a portable AM/FM cassette player, and the Shader Watch-TV, a wrist watch-size black-and –white television. The production process for each product is similar in that both require a certain number of hours of electronic work and a certain number of labour-hours in the assembly department. Each Walkman takes 4 hours of electronic work and 2 hours in the assembly shop. Each Watch-TV requires 3 hours in electronics and 1 hour in assembly shop.

During the current production period, 240 hours of electronic time are available and 100 hours of assembly department time are available. Each Walkman sold yields a profit $7 and each Watch-TV produced may be sold for a profit of $5. Shader's problem is to determine the best possible combination of Walkmans and Watch-TV to manufacture in order to reach the maximum product.

i. State the objective function in mathematical terms.
ii. State the constraint inequalities.
iii. Graph these constraints on a suitable graph and identify the feasible region.
iv. Advise the company on the optimal product mix and contribution.
v. Conduct a sensitivity analysis to find out how the optimum solution will respond to changes in the right-hand –side values of the constraints.
vi. Find the dual prices.

<u>QUESTION 7</u>.

A company operates two types of aircraft, the RS101 and the JC111. The RS101 is capable of carrying 40 passengers and 30 tons of cargo, whereas the JC111 is capable of carrying 60 passengers and 15 tons of cargo. The company is contracted to carry at least 480 passengers and 180 tons of cargo each day.

i. If the cost per journey is $500 for a RS101 and $600 for a JC111, what choice of aircraft will minimize operation cost?
ii. Conduct a sensitivity analysis to find out how the optimum solution will respond to changes in the right-hand –side values of the constraints.
iii. Find the dual prices.

QUESTION 8

A chemical manufacturer processes two chemicals X and Y, in varying proportions to produce 3 products, A, B, and C. He wishes to produce at least 150 units of A, 200 units of B, and 60 units of C. Each ton of X yields 3 of A, 5 of B and 3 of C. Each ton of Y yields 5 of A, 5 of B and 1 of C.

If X costs $40 per ton and Y costs $50 per ton:

iv. State the objective function in mathematical terms.

v. State the constraint inequalities.

vi. Graph these constraints on a suitable graph and identify the feasible region.

vii. Advise the manufacturer on how to minimize cost.

viii. Find the dual prices and interpret the results.

ix. Conduct a sensitivity analysis to find out how the optimum solution will respond to changes in the right-hand –side values of the constraints.

QUESTION 9

LawnGrow Manufacturing Company must determine the mix of its commercial riding mower products to be produced next year. The company produces two product lines, the Max and the Multimax. The average profit is $400 for each Max and $800 for each Multimax. Fabrication and assembly are limited resources. There is a maximum of 5,000 hours of fabrication capacity available per month (Each Max requires 3 hours and each Multimax requires 5 hours). There is a maximum of 3,000 hours of assembly capacity available per month (Each Max requires 1 hour and each Multimax requires 4 hours). Question: How many of each riding mower should be produced each month in order to maximize profit?