UNIT 2

TRANSPORTATION PROBLEM

OUTLINE

Session 2.1: Introduction

Session 2.2: Terminologies

Session 2.3: Transportation problem

Session 2.4: Balancing a transportation problem

Session 2.5: Initial feasible solution

Session 2.6: Finding the optimum solution

OBJECTIVES

By the end of this unit, you should be able to:

1. Identify a transportation problem.
2. Construct a balanced transportation tableau.
3. Find initial and optimum solutions to transportation problem

Note: In order to achieve these objectives, you need to spend a minimum of three(3) hours and a maximum of four (4) hours working through the sessions.

SESSION 2.1: INTRODUCTION

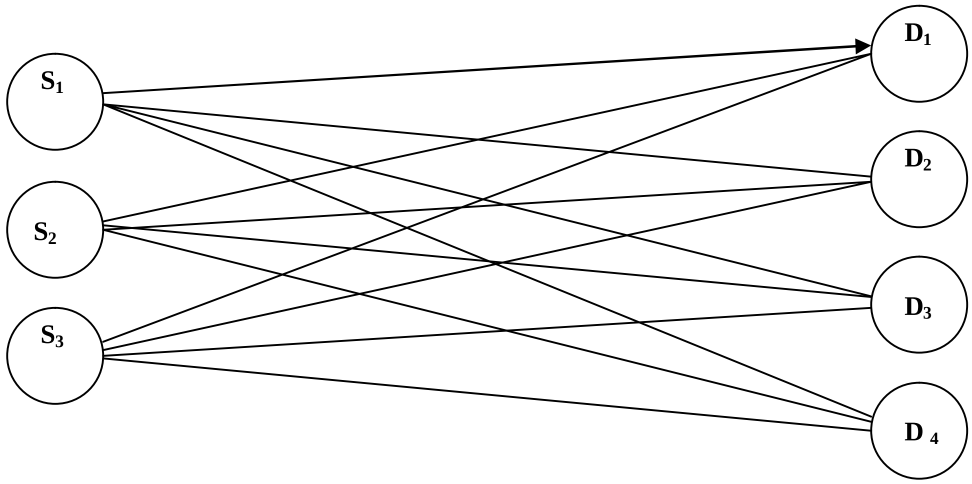
The general transportation problem deals with distributing any commodity from any group of supply centers called source to any group of receiving centers called destination in such a way as to minimize total distribution cost.

Thus, in general, source i (i =1,2,3,..,m) has supply of s_i units to distribute and destination j(j=1, 2, 3, …,n) has a demand for d_j units to be received from the sources. A basic assumption is that the cost of distributing units from source i to destination j is directly proportional to the number distributed, where c_{ij} denotes the cost per unit distributed.

SESSION 2.2: TERMINOLOGIES:

1. **Source:** A location at which a supply of the commodity is available. The sources are numbered from 1 to m.
2. **Destination:** A location at which there is demand for the commodity. The destinations are numbered from 1 to n
3. **Supply (S_i):** The amount of the commodity required at source i (i = 1, 2, 3, 4, 5, …, m).
4. **Demand (d_j):** The amount of the commodity required at destination j (j = 1, 2,,3, 4, 5, …, n).
5. **Transportation cost (C_{ij}):** The cost of transporting a single unit of the product from source i(i = 1, 2, 3, 4, 5, …, m) to destination j(j = 1, 2,,3, 4, 5, …, n). The unit cost is independent of the quantity shipped, so total cost function is linear.
6. **Distribution variables (X_{ij}):** The amount shipped from source i(i = 1, 2, 3, 4, 5, …, m) to destination j(j = 1, 2,,3, 4, 5, …, n). These are the decision variables of the problem.
7. **Transportation Tableau:** A table format used for hand calculations to solve the transportation problem with the simplex method.

The figure below depicts the transportation model or network described above.



A source or destination is represented by a node. The arc joining a source and a destination represents route through which the commodity is transported.

SESSION 2.3: TRANSPORTATION PROBLEM

A transportation problem is specified by:

- The supply
- The demand
- The shipping cost.

Hence the relevant data for any transportation problem can be summarised in a transportation tableau as shown below.

FROM	**TO**				
	D_1	D_2	**…..**	D_n	**Supply**
S_1	C11	C12	…	C1n	s_1
S_2	C21	C22	…	C2n	s_2
S_3	C31	C32	…	C3n	s_3
S_m	Cm1	Cm2	…	Cmn	s_m
Demand	D_1	D_2			D_n

The square or cell in row i and column j of a transportation tableau corresponds to the variable X_{ij}. If X_{ij} is a basic variable(solution), its value is placed in the center of the C_{ij}

cell of the tableau. The tableau format implicitly expresses the supply and demand constraints through the fact that the sum of the variables in row i must equal s_i and the sum of the variables in column j must equal d_j

ILLUSTRATION: Powerco has 3 electric power plants that supply the power needs of 4 cities. Each power plant supply the following number of kilowatt-hour (Kwh) of electricity:

Plant 1 = 35million

Plant 2 = 50million

Plant 3 = 40 million

The peak power demands are as follows:

City 2 = 45 million

City 2 = 20 million

City 3 = 30 million

City 4 = 30million

The costs in dollars ($) of sending 1 million Kwh of electricity from plant to city depends on distance and are shown in the table below:

	TO			
FROM	CITY 1	CITY 2	CITY 3	CITY 4
PLANT 1	8	6	10	9
PLANT 2	9	12	13	7
PLANT 3	14	9	11	5

A transportation problem is specified by the following information:

- A set of m supply points from which a good is shipped. Supply point i can supply at most Si units. In the Powerco example, m=3, S_1 =35, S_2 =50, and S_3 =40
- A set of demand points to which the good is shipped. Demand point j can receive at least dj units of the shipped good. In the Powerco example, n = 4, d_1 = 45, d_2 =20, d_3=30, and d_4=30.
- Each unit produced at supply point i and shipped to demand point j incurs a variable cost of C_{ij}. In the Powerco example C_{12}=6, C_{34} =5,etc.

In the Powerco example, total supply and total demand equal 125, so the Powerco problem is an example of a balanced transportation problem.

Management's objective is to determine the routs to be used and the quantities to be transported via each route that will provide the minimum total transportation cost. The cost for transporting each unit is given in the table above. Because the objective of the transportation problem is to minimize the total transportation cost, the cost data in the table above is used to formulate the objective function. The transportation problems need constraints because each origin has a limited supply and each destination has a specific demand. Data from the source and destination are used to formulate the constraints for the transportation problem. Combining the objective function and all constraints into one model provides a 12-variable, 7-constraint linear programming model.

SESSION 2.4: BALANCING A TRANSPORTATION PROBLEM

It should be noted that the transportation model has feasible solution only if $\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$ (that is total supply equal total demand). If this condition exits, the problem is said to be a balanced transportation problem, otherwise it is unbalanced.

If the problem has physical significance and this condition is not met, it usually means that either S_i or D_j actually represents a bound rather than an exact requirement. If this is the case, a fictitious 'source' or 'destination' (called the dummy source or the dummy destination) can be introduced to take up the slack in order to convert the inequalities into equalities and satisfy feasibility conditions.

1. **If total supply exceeds total demand:** If this situation occurs, the modification is to add an extra column (dummy destination) to the tableau with the demand for the dummy destination equal to the excess supply. In other words, we balance the transportation problem by creating a dummy demand point that has a demand equal to the amount of excess supply. Since shipments to the dummy points are

not real shipments, they are assigned a cost of zero (0). Shipments to the dummy demand point indicate unused supply capacity.

2. **If total supply is less than total demand:** This requires a dummy source with supply equal to the excess of demand over supply. In other words, we add a dummy source that will absorb the difference (excess of supply over demand)

Illustration: A company has plants in locations A, B and C. Its major distribution centres are located in P and Q. The capacities of the 3 plants during the next quarter are 1000, 1500, and 1200 cars. The quarterly demands of the two distribution centres are 2300 and 1400 cars. The transportation costs per kilometer are shown below.

TO

FROM	P	Q
A	80	215
B	100	108
C	102	68

❖ The balanced transportation tableau is shown below.

DESTINATION

SOURCE	P	Q	SUPPLY
A	80	215	1000
B	100	108	1500
C	102	68	1200
DEMAND	2300	1400	3700

❖ The transportation tableau above is balanced because:
total supply.(1000+1500+1200=3700) equals total demand (2300+1400=3700)

❖ Suppose the plant B' capacity is 1300 instead of 1500, the situation is said to be unbalanced because total supply (1000+1300+1200=3500) does not equal total demand (2300+1400=3700).

- Our objective is to reformulate the transportation model in a manner that will distribute the shortage quantity (3700-3500=200) optimally among the distribution centres.
- Since demand exceeds supply, a dummy source (plant) can be added with the capacity equal to 200 cars.
- This is shown in the diagram below.

DESTINATION

SOURCE	P	Q	SUPPLY
A	80	215	1000
B	100	108	1500
C	102	68	1200
Dummy Plant	0	0	200
DEMAND	2300	1400	3700

- In a similar manner, if the supply exceeds the demand, we can add a dummy destination that will absorb the difference. Suppose demand at P drops to 1900 instead of 2300, the situation is said to be unbalanced because total supply (1000+1500+1200=3700) does not equal total demand (1900+1400=3300). The balanced transportation tableau is shown below.

DESTINATION

SOURCE	P	Q	Dummy	SUPPLY
A	80	215	0	1000
B	100	108	0	1500
C	102	68	0	1200
DEMAND	1900	1400	400	3700

- ❖ Any cars shipped from a plant to a dummy destination centre represent a surplus quantity at that plant. The associated unit transportation cost is zero (0).

SESSION 2.5: FINDING THE INITIAL FEASIBLE SOLUTION

A feasible solution of a transportation problem is a set of entries, which satisfy the following conditions:

- The entries must be non-negative, since negative shipments are not acceptable
- The entries must sum along each row to the capacity available at that factory and down each column to the requirement of that warehouse.

The total transportation cost of any solution is obtained by multiplying each unit cost and summing overall entries. The object is to find the feasible solution(s) with the lowest total cost. This is called the optimum solution

Steps to follow:

1. Construct a transportation tableau
2. Use the least cost first rule to allocate to the cells

EXAMPLE: Suppose a company has factories at Manchester, Birmingham, and London, and warehouses at Miniopolis, Coventry, Boston, and Cardiff. For the coming planning period the factories and warehouses have capacities and requirements set as follows:

CAPACITIES AND REQUIREMENTS

FACTORY CAPACITY (`000 UNITS)		WAREHOUSE REQUIREMENTS (`000 UNITS)	
Manchester	20	Miniopolis	11
Birmingham	10	Coventry	13
London	25	Boston	17
		Cardiff	14
Total supply	55	Total demand	55

Per unit transportation costs by the cheapest mode of transportation currently available are given in the table below.

TO

FROM	Minneapolis	Coventry	Boston	Cardiff
Manchester	1	6	3	6
Birmingham	7	3	1	6
London	9	4	5	4

Solution

Step 1: Laying out the problem:

It is convenient to set out the basic information in a simple table called the **Transportation Tableau.** There is one row for each origin (factory) and one column for each destination (warehouse). Consequently, there is one cell for each possible shipment of goods. Formalize the problem requirements by checking if $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. This is done to find out if it is necessary to introduce a dummy. Since $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$, there will not be the need for a dummy. The balanced transportation tableau is shown below.

DESTINATION

ORIGIN	Minneapolis	Coventry	Boston	Cardiff	**Capacities**
Manchester	1	6	3	6	**20**
Birmingham	7	3	1	6	**10**
London	9	4	5	4	**25**
Requirements	**11**	**13**	**17**	**14**	**55**

Step 2: Finding the initial feasible solution:

The method of "Least – cost first" rule: Put as many as possible in the cell with the lowest unit cost, then in the cell with the second lowest unit cost, and so on. In the case of a tie, arbitrarily pick any of the tied cells.

- In the above example, one would begin by shipping eleven (11) units from the factory in Manchester to a warehouse in Minneapolis. (Eleven to Minneapolis because the total requirement is 11)
- Ten (10) units from factory in Birmingham to warehouse in Boston (Ten from Birmingham because the total "i.e.- maximum" there is 10)
- Seven (7) unit from factory in Manchester to warehouse in Boston (Seven to Boston because the total requirement is 17. It had 10 already from Birmingham).
- Fourteen (14) from factory in London to warehouse in Cardiff (Fourteen to Cardiff because the requirements for Cardiff is 14).
- Eleven (11) from factory in London to warehouse in Coventry (Eleven from London because total capacity for London is 25. It had already shipped 14 to Cardiff).
- Two (2) from factory in Manchester to warehouse in Coventry.(Two to Coventry because total requirements for Coventry is 13. She already had 11 from London or two from Manchester because total capacity for Manchester is 20, and 18 had already been shipped out: - Eleven (11) to Minneapolis and seven (7) to Boston.

The distribution is shown in the table below.

DESTINATION

ORIGIN	Min.	Cov.	Bost.	Card.	CAPACITIES
Manchester	1 **11**	6 **2**	3 **7**	6	**20**
Birmingham	7	3	1 **10**	6	**10**
London	9	4 **11**	5	4 **14**	**25**
REQUIREMENTS	**11**	**13**	**17**	**14**	**55**

Total cost = $1 \times 11 + 6 \times 2 + 3 \times 7 + 1 \times 10 + 4 \times 11 + 4 \times 14$

= 11 + 12 + 21 + 10 + 44 + 56

= <u>$154.00</u>

SESSION: 2.6: FINDING THE OPTIMUM SOLUTION

There is the need to check to see if the **initial feasible solution** is the optimum cost. This is done by calculating what are known as "shadow cost" and comparing these with the actual costs to see whether a change of allocation is desirable. We start by calculating "dispatch" and "reception" costs for each used cell. It is assumed that the transportation cost in each cell can be split into two costs: dispatch" and "reception. The dispatch costs are denoted by U_i and the reception costs are denoted by V_j.

Thus a feasible solution is optimal if and only if (C_{ij} - u_i - v_j) ≥ 0 for every (i , j) such that X_{ij} in the unused cells. The only work required by the optimality test is the derivation of the values of u_i and v_j for the used cells and then the calculation of (C_{ij} - u_i - v_j) for the unused cells.

Since (C_{ij} - u_i - v_j) is required to be zero if X_{ij} is in a used cell, the d_j and S_i satisfy the set of equation $C_{ij} = d_j + S_j$ for each (i, j) such that X_{ij} is in a used cell. If (C_{ij} - u_i - v_j) ≥ 0, then optimality is reached. Otherwise, we conclude that the current feasible solution is not optimal. Therefore the transportation simplex method must go to the iteration step to find a better feasible solution.

The example below will be used to illustrate the optimum solution.

EXAMPLE 1: A company has three factories (A1, A2, and A3) and four distribution warehouses (B1, B2, B3, and B4). It is required to schedule factory shipments from factories to warehouses in such a manner as to minimize the total cost of shipping. Below are the factory capacities and warehouse requirements.

CAPACITIES AND REQUIREMENTS

FACTORY CAPACITY		WAREHOUSE REQUIREMENTS	
A1	20	B1	40
A2	60	B2	10
A3	70	B3	60
		B4	40
Total supply	150	Total demand	150

The per unit costs of transporting from factory to warehouse are also shown below.

DESTINATION

ORIGIN	B1	B2	B3	B4
A1	6	16	4	2
A2	2	8	6	10
A3	14	4	2	12

<u>Solution:</u>

Step 1. Lay out the problem:

Formalize the problem requirements by checking if $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. This is done to find out if it is necessary to introduce a dummy. Here, $\sum_{i=1}^{m} s_i =$ 20+60+70=150, and $\sum_{j=1}^{n} d_j$ =40+10+60+40=150. Since $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$, there is no need to introduce dummy in this example. The balanced transportation tableau is shown below.

DESTINATION

ORIGIN	B1	B2	B3	B4	CAPACITIES
A1	6	16	4	2	**20**
A2	2	8	6	10	**60**
A3	14	4	2	12	**70**
REQUIREMENTS	**40**	**10**	**60**	**40**	**150**

Step 2. Find the initial feasible solution: The 'Least –cost-first' rule is used.

DESTINATION

ORIGIN	B1	B2	B3	B4	CAPACITIES
A1	6	16	4	2 **20**	**20**
A2	2 **40**	8	6 **20**	10	**60**
A3	14	4 **10**	2 **40**	12 **20**	**70**
REQUIREMENTS	**40**	**10**	**60**	**40**	**150**

Initial cost = $2\times 20+2\times 40+6\times 20+4\times 10+2\times 40+12\times 20=\$600.00$

Step 3: Optimality test

<u>**Procedure:**</u>

1. Find equations in terms of U's and V's using the used cells.
2. Solve for the U's and V's. by convention, the dispatch cost from the first is set to zero(U_1= 0).
3. Find the difference between the shadow costs and actual costs for the <u>**unused cells**</u> using the following formula: $C_{ij}-(U_i+V_j)$.

Note: The shadow cost is given by U_i+V_j. This is an imputed cost of not using a particular cell(route).

A. Equations in terms of U's and V's for the used cells

X_{14} : $U_1+V_4 = 2$……………….(1)

X_{21} : $U_2+V_1 = 2$……………,....(2)

X_{23} : $U_2+V_3 = 6$……………….(3)

X_{32} : $U_3+V_2 = 4$……………….(4)

X_{33} : $U_3+V_3 = 2$……………….(5)

X_{34} : $U_3+V_4 = 12$……………...(6)

B. Solving for the U's and V's for in A above

NOTE: Let U_1= 0

From (1): $U_1+V_4 = 2$; $\Rightarrow V_4 = 2 - 0$ $\therefore$ $\mathbf{V_4 = 2}$ ....(7)

From (6): $U_3+V_4=12$(But $V_4=2$, from 7); $\Rightarrow U_3 =12 - 2$ $\therefore$ $\mathbf{U_3 = 10}$ ....(8)

From (4): $U_3+V_2=4$(But $U_3=10$, from 8); $\Rightarrow V_2 =4 - 10$ $\therefore$ $\mathbf{V_2 = -6}$ ....(9)

From (5): $U_3+V_3=2$(But $U_3=10$, from 8); $\Rightarrow V_3 =2 - 10$ $\therefore$ $\mathbf{V_3 = -8}$ ....(10)

From (3): $U_2+V_3=6$(But $V_3=-8$, from 10); $\Rightarrow U_2 =6 - (-8)$ $\therefore$ $\mathbf{U_2 = 14}$ ....(11)

From (2): $U_2+V_1=2$(But $U_2= 14$, from 11); $\Rightarrow U_2 =2 - 14$ $\therefore$ $\mathbf{V_1 = -12}$

C. Find $C_{ij} - (U_i + V_j)$ for every (i , j) for the unused cells

Note: Use the values of the U's and V's above

$X_{11} = C_{11} - U_1 - V_1 = 6 - 0 - (-12) = 18$

$X_{12} = 16 - 0 - (-6) = 22$

$X_{13} = 4 - 0 - (-8) = 12$

$X_{22} = 8 - 14 - (-6) = 0$

$X_{24} = 10 - 14 - 2 = -6$

$X_{31} = 14 - 10 - (-12) = 16$

Since $X_{24} = -6$, which is a negative number, we conclude that the current basic feasible solution is not optimal. Therefore, the transportation simplex method must go to the iteration step to find a better feasible solution

Iteration step:

i. Determine the entering basic variable. Select the nonbasic variable X_{ij} having the largest (in absolute terms) negative value of $C_{ij} - (U_i + V_j)$

ii. Determine the leaving basic variable. Identify the chain reaction required to retain feasibility when the entering basic variable is increased. From among the donor cells, select the basic variable having the smallest value.

iii. Determine the new feasible solution. Add the value of the leaving basic variable to the allocation for each recipient cell. Subtract this value from the allocation for each donor cell.

<u>Step 1</u>: The entering basic variable. Select the nonbasic variable X_{ij} having the largest (in absolute terms) negative value of $C_{ij} - (U_i + V_j)$ is $X_{23.}$ It has a value of -6.

Part 1: We have to determine an entering basic variable. Since $C_{ij} - (U_i + V_j)$ represents the rate at which the objective function would change as the nonbasic variable X_i is increased, the entering basic variable must have a negative $C_{ij} - (U_i + V_j)$ to decrease the total cost Z. In cases where there are more than one negative values, the value having the largest (in absolute terms) negative value of $C_{ij} - (U_i + V_j)$ is chosen to be the entering basic variable. In our example, X_{24} is chosen.

Part 2: Increasing the entering basic variable from zero sets off a chain reaction of compensating changes in other basic variables (allocations) in order to continue satisfying the supply and demand constraints. The first basic variable to be decreased to zero first then becomes the leaving basic variable.

With X_{24} as the entering basic variable, the chain reaction is relatively a simple one. We shall always indicate the entering basic variable by placing a + sign in its cell. Thus, increasing X_{24} requires decreasing X_{23} by the same amount to restore the supply of 60 in row 2, which in turn requires increasing X_{33} by this amount to restore the demand of 60 in column 3., which in turn requires decreasing X_{34} by this amount to restore the supply of 70 in row 3. This decrease in X_{34} successfully completes he chain reaction because it also restores the demand in column 4. The net result is that cell(2, 4) and cell(3, 30 become recipient cells, each receiving its additional allocation from the donor cells, (2, 30 and (3, 4). These cells are indicated by the + and – signs.

Each donor cell decreases its allocation by exactly the same amount that the entering basic variable is increased. Therefore the donor cell that starts with the smallest allocation must reach a zero allocation first as the entering basic variable X_{24} is increased. In our example, any of X_{23} and X_{34} becomes the leaving basic variable.

In general, there always is just one chain reaction (in either direction) that can be completed successfully to maintain feasibility when entering basic variables is increased from zero

Part 3: The new basic feasible solution is identified simply by adding the value of the leaving basic variable (before any change) to the allocation for each recipient cell and subtracting this same amount from the allocation for each donor cell. In our example, the value of the leaving basic variable is 20(X_{34}), so this portion of the transportation simplex tableau changes as shown in the table below for the new solution. Since X_{34} is nonbasic in the new solution, its new allocation of zero is no longer shown in the new tableau.

Test for optimality, and if the current solution is optimal, stop. Otherwise, go to the iteration step.

DESTINATION

ORIGIN	**B1**	**B2**	**B3**	**B4**	**CAPACITIES**
A1	6	16	4	2 **20**	**20**
A2	2 **40**	8	6 **20(-)**	10 (+)	**60**
A3	14	4 **10**	2 **40**(+)	12 **20(-)**	**70**
REQUIREMENTS	**40**	**10**	**60**	**40**	**150**

Step 2

DESTINATION

ORIGIN	**B1**	**B2**	**B3**	**B4**	**CAPACITIES**
A1	6	16	4	2 **20**	**20**
A2	2 **40**	8	6 **20(-)20**	10 (+)20	**60**
A3	14	4 **10**	2 **40**(+)20	12 **20(-)20**	**70**
REQUIREMENTS	**40**	**10**	**60**	**40**	**150**

Step 3

DESTINATION

ORIGIN	**B1**	**B2**	**B3**	**B4**	**CAPACITIES**
A1	6	16	4	2 **20**	**20**
A2	2 **40**	8	6	10 **20**	**60**
A3	14	4 **10**	2 **60**	12	**70**
REQUIREMENTS	**40**	**10**	**60**	**40**	**150**

Optimum cost = $2\times 20+2\times 40+10\times 20+4\times 10+2\times 60=\$480.00$

WORKED EXAMPLE 1

A firm of wholesale domestic suppliers, with 3 warehouses, received orders for a total of 100 deep freezers from 4 retail shops. In total in the 3 warehouses, there are 110 deep freezers available and the management wishes to minimize transport cost by dispatching the freezers required from the appropriate warehouses.

Details of availabilities in the warehouses and requirements of the shops are given in the table below.

AVAILABILITY		REQUIREMENTS	
Warehouse 1	40	Shop A	25
Warehouse 2	20	Shop B	25
Warehouse 3	50	Shop C	42
	110	Shop D	8
TOTAL	110	TOTAL	100

Per unit transportation cost (in thousands of dollars) by the cheapest mode of transportation currently available are given in the table below.

	TO			
FROM	SHOP A	SHOP B	SHOP C	SHOP D
Warehouse1	3	16	9	2
Warehouse2	1	9	3	8
Warehouse3	4	5	2	5
TOTAL				

i. Construct a balanced transportation tableau for the problem above.

ii. Determine the minimum cost of transporting deep freezers from the warehouses to the shops.

<u>Solution</u>

Step 1. Laying out the problem:

<u>TO</u>

<u>FROM</u>	Shop A	Shop B	Shop C	Shop D	Dummy	Available
Warehouse I	3	16	9	2	0	40
Warehouse II	1	9	3	8	0	20
Warehouse III	4	5	2	5	0	50
Requirements	25	25	42	8	10	110

Step 2. Find the initial feasible solution: Using the 'Least –cost-first' rule.

<u>TO</u>

<u>FROM</u>	Shop A	Shop B	Shop C	Shop D	Dummy	Available
Warehouse I	3 5	16 17	9	2 8	0 10	40
Warehouse II	1 20	9	3	8	0	20
Warehouse III	4	5 5	2 42	5	0	50
Requirements	25	25	42	8	10	110

<u>(ii).Cost of allocation from warehouses to shops</u>

Warehouse I to Shop A	5 units @ $ 3	= $15
Warehouse I to Shop B	17 units @ $ 16	= $272
Warehouse I to Shop D	8 units @ $ 2	= $16
Warehouse I to Shop Dummy	10units @ $0	=0
Warehouse II to Shop A	20 units @ $ 1	= $20
Warehouse III to shop B	8 units @ $ 5	= $40
Warehouse III to shop C	42 units @ $ 2	= $84
Total		$447.00

Step 3:Optimality test: The basic feasible solution is optimal if and only if$(C_{ij} - u_i - v_j) \geq 0$ for every (i , j) such that X_{ij} is nonbasic

Occupied cells(Equations in terms Ui 's and Vj 's for each basic variable Xij(cells with allocation or used cells)

X_{11}: $U_1+V_1=3$……………….(1)

X_{12}: $U_1+V_2=16$…………….,..(2)

X_{14}: $U_1+V_4=2$……………….(3)

X_{15}: $U_1+V_5=0$……………….(4)

X_{21}: $U_2+V_1=1$……………….(5)

X_{32}: $U_3+V_2=5$……………….(6)

X_{33}: U_3+V_32……………….(7)

Solving for Ui's and Vj's

Put $U_1 = 0$

From (1): $0 + V_1 = 3 \Rightarrow V_1 = 3 - 0, \therefore V_1 = 3$

From (2): $0+V_2 =16 \Rightarrow V_2 = 16 - 0, \therefore V_1 = 16$

Substituting, the remaining U's and V's are obtained(please check)

$V_3 = 13$

$V_4 = 2$

$V_5 = 0$

$U_1 = 0$

$U_2 = -2U_3 = -11$

Calculating shadow costs of the unused roots:

$C_{13} = U_1 + V_3 = 0 + 13 = 13$

$C_{22} = U_2 + V_2 = -2 + 16 = 14$

$C_{23} = U_2 + V_3 = -2 + 13 = 11$

$C_{24} = U_2 + V_4 = -2 + 2 = 0$

$C_{25} = U_2 + V_5 = -2 + 0 = -2$

$C_{31} = U_3 + V_1 = -11 + 3 = -8$

$C_{35} = U_3 + V_5 = -11 + 0 = -11$

Calculating the difference between actual costs and shadow costs

Calculating the difference between actual costs and shadow costs

Cell	Actual cost	Shadow cost	Difference
Warehouse I to shop C	9	13	-4
Warehouse II to shop B	9	14	-5
Warehouse II to shop C	3	11	-8
Warehouse II to shop D	8	0	8
Warehouse II to shop Dummy	0	-2	2
Warehouse III to shop A	4	-8	12
Warehouse III to shop D	5	-9	14
Warehouse III to shop Dummy	0	-11	11

These computed ‘shadow costs’ are compared with the actual transport costs. Where actual costs are less than the shadow costs, overall costs can be reduced by allocating units into that cell. It means that the current solution is not optimum.

Reallocation

	Shop A	Shop B	Shop C	Shop D	Dummy	Available
Warehouse I	**5+**	**17-**		**8**	**10**	40
Warehouse II	**20-**		+			20
Warehouse III		**18-**	**42-**			50
Requirements	25	25	42	8	10	110

Optimum allocation

	Shop A	Shop B	Shop C	Shop D	Dummy	Available
Warehouse I	**22**			**8**	**10**	40
Warehouse II	**3**		**17**			20
Warehouse III		**25**	**25**			50
Requirements	25	25	42	8	10	110

The minimum cost of allocation is

22x3 + 8x2 + 10x0 + 3x1 + 17x3 + 25x5 + 25x2 = $311.

WORKED EXAMPLE 2: Refer to the Powerco problem

A. Equations in terms of U's and V's for the used cells

X_{11} : $U_1+V_1= 8$……………….(1)

X_{12} : $U_1+V_2 = 6$……………,....(2)

X_{21} : $U_2+V_1 = 9$……………….(3)

X_{23} : $U_2+V_3 = 13$……………….(4)

X_{33} : $U_3+V_3 = 11$……………..….(5)

X_{34} : $U_3+V_4 = 5$……………….(6)

B. Solving for the U's and V's for in A above

NOTE: Let $U_1 = 0$

From (1): $U_1+V_1= 8$; $\Rightarrow V_4 = 8 - 0$ $\therefore$ $\mathbf{V_1 = 8}$ ....(7)

From (2): $U_1+V_2= 6$(But $U_1=0$); $\Rightarrow V_2 =6 - 0$ $\therefore$ $\mathbf{V_2 = 6}$ ....(8)

From (3): $U_2+V_1=9$(But $V_1= 8$, from 7); $\Rightarrow U_2 =9 - 8$ $\therefore$ $\mathbf{U_2 = 1}$ ....(9)

From (4): $U_2+V_3=13$(But $U_2=1$, from 9); $\Rightarrow V_3 =1 - 10$ $\therefore$ $\mathbf{V_3 = 12}$ ....(10)

From (5): $U_3+V_3=11$(But $V_3=12$, from 10); $\Rightarrow U_3 =11 – 12$ $\therefore$ $\mathbf{U_3 = -1}$......................(11)

From (6): $U_3+V_4=5$ (But $U_3= -1$, from 11); $\Rightarrow V_4 =5 – (-1)$ $\therefore$ $\mathbf{V_4 = 6}$

C. Find (C_{ij} - u_i - v_j) for every (i , j) for the unused cells

Note: Use the values of the U's and V's above

$X_{13} = C_{13} – U_1 – V_3 = 10- 0-12= - 2$

$X_{14} = C_{14} – U_1 – V_4 = 9 – 0 – 6 = 3$

$X_{22} = C_{22} – U_2 – V_2 = 12 – 1 – 6 = 5$

$X_{24} = C_{24} – U_2 – V_4 = 7 – 1 – 6 = 0$

$X_{31} = C_{31} – U_3 – V_1 = 14 – (-1) – 8 = 7$

$X_{32} = C_{32} – U_3 – V_2 = 9 – (-1) –6 = 4$

Since $X_{13} = -2$, which is a negative number, we conclude that the current basic feasible is not the optimum. Move to the iteration step.

Balanced Tableau for the Powerco problem

		TO			
FROM	CITY 1	CITY 2	CITY 3	CITY 4	**Supply**
PLANT 1	8	6	10	9	**35**
PLANT 2	9	12	13	7	**50**
PLANT 3	14	9	11	5	**40**
Demand	**45**	**20**	**30**	**30**	

The Powerco problem and its initial solution

		TO			
FROM	CITY 1	CITY 2	CITY 3	CITY 4	**Supply**
PLANT 1	8 15	6 20	10	9	**35**
PLANT 2	9 30	12	13 20	7	**50**
PLANT 3	14	9	11 10	5 30	**40**
Demand	**45**	**20**	**30**	**30**	**125**

Reallocation

		TO			
FROM	CITY 1	CITY 2	CITY 3	CITY 4	**Supply**
PLANT 1	8 - 15	6 20	10 +	9	**35**
PLANT 2	9 30 +	12	13 20 -	7	**50**
PLANT 3	14	9	11 10	5 30	**40**
Demand	**45**	**20**	**30**	**30**	**125**

Optimum solution

		TO			
FROM	CITY 1	CITY 2	CITY 3	CITY 4	**Supply**
PLANT 1	8	6 20	10 15	9	**35**
PLANT 2	9 45	12	13 5	7	**50**
PLANT 3	14	9	11 10	5 30	**40**
Demand	**45**	**20**	**30**	**30**	**125**

Optimum transportation cost = 6x20 + 10x15 + 9x45 + 13x5 + 11x10 + 5x30 =$1000.00

SELF ASSESSMENT QUESTIONS

QUESTION 1

Two reservoirs are available to supply the water needs of three cities. Each reservoir can supply up to 50 million gallons of water per day. Each city would like to receive 40 million gallons of water per day. For each million gallons per day of unmet demand, there is a penalty. At city 1, the penalty is $20; at city 2, the penalty is $22; and at city 3, the penalty is $23. The costs of transporting 1 million gallons of water from each reservoir to each city are shown in the table below.

	TO		
FROM	City 1	City 2	City 3
Reservoir 1	$7	$8	$10
Reservoir 2	$9	$7	$8

Formulate a balanced transportation problem that can be used to minimize the sum of shortages and transport costs.

QUESTION 2

A company has plants in Los Angeles, Detriot and New Orleans. Its major distribution centres are located in Denver and Miami. The capacities of the 3 plants during the next quarter are 1000,1500, and 1200 cars. The quarterly demand of the two distribution centres is 2300 and 1400 cars. The transportation cost per kilometer is shown below.

	TO	
FROM	DENVER	MIAMI
LOS ANGELES	80	215
DETRIOT	100	108
NEW ORLEANS	102	68

Formulate a balanced transportation problem that can be used to minimize the sum of shortages and transport costs.

QUESTION 3

A company has three factories (A1, A2, and A3) and four distribution warehouses (B1, B2, B3, and B4). It is required to schedule factory shipments from factories to warehouses in such a manner as to minimize the total cost of shipping. Below are the factory capacities, warehouse requirements and the shipping cost in thousands of cedis per unit.

UNIT TRANSPORT COSTS					
	Warehouse B1	Warehouse B2	Warehouse B3	Warehouse B4	Capacity
Factory A1	4	6	8	3	700
Factory A2	5	5	7	4	900
Factory A3	2	3	11	9	1600
Requirement	800	500	600	1300	3200

i. State with reason(s) whether or not the above transportation problem is balanced.

ii. If no, describe how it will be balanced. Illustrate with a diagram.(s).

QUESTION 4

Suppose a company has factories at Manchester, Birmingham, and London, and warehouses at Miniopolis, Coventry, Boston, and Cardiff. For the coming planning period the factories and warehouses have capacities and requirements set as follows:

CAPACITIES AND REQUIREMENTS

FACTORY CAPACITY (`000 UNITS)		WAREHOUSE REQUIREMENTS (`000 UNITS)	
Manchester	20	Miniopolis	11
Birmingham	10	Coventry	13
London	25	Boston	17
		Cardiff	14
Total supply	55	Total demand	55

Per unit transportation costs by the cheapest mode of transportation currently available are given in the table below.

TO

FROM	Miniopolis	Coventry	Boston	Cardiff
Manchester	1	6	3	6
Birmingham	7	3	1	6
London	9	4	5	4

Formulate a balanced transportation problem that can be used to minimize the sum of shortages and transport costs.

QUESTION 5

A firm of wholesale domestic suppliers, with 3 warehouses, received orders for a total of 15 deep freezers from 4 retail shops. In total in the 3 warehouses, there are 15 deep freezers available and the management wishes to minimize transport cost by dispatching the freezers required from the appropriate warehouses.

Details of availabilities in the warehouses and requirements of the shops are given in the table below.

AVAILABILITY		REQUIREMENTS	
Warehouse X	2	Shop A	3
Warehouse Y	6	Shop B	3
Warehouse Z	7	Shop C	4
		Shop D	5
TOTAL	15	TOTAL	15

Per unit transportation cost (in thousands of dollars) by the cheapest mode of transportation currently available are given in the table below.

TO

FROM	SHOP A	SHOP B	SHOP C	SHOP D
Warehouse	13	11	15	20
Warehouse	17	14	12	13
Warehouse	18	18	15	12

Determine the minimum cost of transporting deep freezers from the warehouses to the shops.

QUESTION 2

Consider the problem faced by the transport department of a medium-sized company that has three factories (A, B, and C) and four warehouses (W, X, Y, and Z). For the coming planning period the factories and warehouses have capacities and requirements set as follows. In addition, the cost of transporting a unit from each factory to each warehouse is also given in the table.

UNIT TRANSPORT COSTS					
	Warehouse W	Warehouse X	Warehouse Y	Warehouse Z	Capacity
Factory A	4	6	8	3	700
Factory B	5	5	7	4	900
Factory C	2	3	11	9	1600
Requirement	800	500	600	1300	3200

Required to determine, which factories should supply which warehouses in such a way as to minimize the total transportation cost.

QUESTION 3

A firm of domestic equipment suppliers, with 3 warehouses (W), received orders for a total 100 deep freezers from 4 retail shops (S). In total in the 3 warehouses there are 110 freezers available and the management wish to minimize transport costs by dispatching the freezers required from the appropriate warehouses. Details of availabilities, requirements, and transport costs are given in the table below.

			Required				
			S1	S2	S3	S4	Total
		Freezers	25	25	42	8	100
	W1	40	$3	$16	$9	$2	Transport cost per freezer
	W2	20	$1	$9	$3	$8	
Available	W3	50	$4	$5	$2	$5	
Total		110					

Required to show how management will achieve their objectives.