UNIT 3

ASSIGNMENT PROBLEM

OUTLINE

Session 2.1: Introduction

Session 2.2: Solution of Minimization Assignment Problem

Session 2.3: Solution of Maximization Assignment Problem

OBJECTIVES

By the end of this unit you should be able to:

1. Identify and explain an Assignment Problem.
2. Solve Minimization and Maximization Assignment Problems.

Note: In order to achieve these objectives, you need to spend a maximum of two (2) hours working through the sessions.

SESSION 3.1: INTRODUCTION

In its most general form, the assignment problem is as follows:

There are a number of *agents* and a number of *tasks*. Any agent can be assigned to perform any task, incurring some *cost* that may vary depending on the agent-task assignment. It is required to perform all tasks by assigning exactly one agent to each task in such a way that the *total cost* of the assignment is minimized.

In the classical assignment problem, the goal is to find an optimal assignment of agents to tasks without assigning an agent more than once and ensuring that all tasks are completed. The objective might be to minimize the total time to complete a set of tasks, or to maximize skill ratings, or to minimize the cost of the assignments. A feature of the assignment problem particular is that only one machine is assigned to one and only one job.

Illustration: A certain machine shop has n machines denoted by M_1, M_2, M_3 …, M_n. A group of n different jobs(J_1, J_2, J_3, …, J_n) is to be assigned to these machines. For each job the machining cost depends on the machine to which it is assigned. Each machine can work only on one job. The problem is to assign the jobs to the machines, which will minimize the total cost of machining.

SESSION 3.2: SOLUTION OF MINIMIZATION ASSIGNMENT PROBLEM

The basic principle is that the optimal assignment is not affected if a constant is added or subtracted from any row or column of the cost matrix. For example if the cost of doing any job in machine 3 in the table below is increased by $2 so that the third row of the cost matrix becomes 4, 3, 3, 4, it can be easily verified that the optimal solution is not affected by this change.

	J_1	J_2	J_3	J_4
M_1	10	9	8	7
M_2	3	4	5	6
M_3	2	1	1	2
M_4	4	3	5	6

In essence, the solution procedure is to subtract a sufficiently large cost from the various rows or columns in such a way that an optimal assignment is found by inspection. We initiate the algorithm by examining each row (column) of the cost matrix to identify the smallest element. This quantity is then subtracted from all the elements in that row (column). This produces a cost matrix containing at least one zero element in each row(column). Now try to make a feasible assignment using the cells with zero costs. If it is possible then we have an optimal assignment.

Examining the rows first, a reduced cost matrix can be obtained by subtracting:

i. Seven(7) from the first row.
ii. Three(3) from the second row.
iii. One(1) from the third row.
iv. Three(3) from the fourth row.

This produces the following results:

	J_1	J_2	J_3	J_4
M_1	3	2	1	0
M_2	0	1	2	3
M_3	1	0	0	1
M_4	1	0	2	3

From the above table a feasible assignment using only the cells with zero costs is

$M_1 \rightarrow J_4$

$M_2 \rightarrow J_1$

$M_3 \rightarrow J_3$

$M_4 \rightarrow J_2$

Hence this is an optimal solution..

In general, it may not always be possible to find a feasible assignment using cells with zero costs.

To illustrate this consider the following assignment problem

	J_1	J_2	J_3	J_4
M_1	10	9	7	8
M_2	5	8	7	7
M_3	5	4	6	5
M_4	2	3	4	5

The solution is as follows.

First, the minimum element in each row is subtracted from all the elements in that row. This gives the following reduced-cost matrix.

	J_1	J_2	J_3	J_4
M_1	3	2	0	1
M_2	0	3	2	2
M_3	1	0	2	1
M_4	0	1	2	3

Since both the machines M_2 and M_4 have a zero cost corresponding to job J_1 only, a feasible assignment using only cells with zero costs is not possible. To get additional zeros subtract the minimum element in the fourth column from all the elements in that column. The result is shown in the table below.

	J_1	J_2	J_3	J_4
M_1	3	2	0	0
M_2	0	3	2	1
M_3	1	0	2	0
M_4	0	1	2	2

Only three jobs can be assigned using the zero cells, so a feasible assignment is still not possible. In such cases, the procedure draws a minimum number of lines through some selected rows and columns in such a way that all the cells with zero costs are covered by these lines. The minimum number of lines needed is equal to the maximum number of jobs that can be assigned using the zero cells.

In the current example, this can be done with three lines as follows.

	J_1	J_2	J_3	J_4
M_1	3	2	0	0
M_2	0	3	2	1
M_3	1	0	2	0
M_4	0	1	2	2

Now select the smallest element, which is not covered by the lines. In the current example, it is 1. Subtract this number from all the elements, which are not covered. Then add this number to all those covered elements that are at the intersection of two lines. This gives the following reduced cost matrix.

	J_1	J_2	J_3	J_4
M_1	4	2	0	0
M_2	0	2	1	0
M_3	2	0	2	0
M_4	0	0	1	1

A feasible assignment is now possible and an optimal solution is assigned

$M_1 \rightarrow J_3$

$M_2 \rightarrow J_1$

$M_3 \rightarrow J_4$

$M_4 \rightarrow J_2$

The total cost is given by: 7 +5+ 5+ 3 = 20

An alternate optimal solution is:

$M_1 \rightarrow J_3$

$M_2 \rightarrow J_4$

$M_3 \rightarrow J_2$

$M_4 \rightarrow J_1$

In case a feasible set could not be obtained at this step, one has to repeat the step of drawing lines to cover the zeros and continue until a feasible assignment is obtained.

SESSION 3.3: SOLUTION OF MAXIMIZATION ASSIGNMENT PROBLEM

The following example will be used to illustrate the maximization algorithm. The figures relate to contribution and it is required to maximize contribution

	W	X	Y	Z
M_1	25	18	23	14
M_2	38	15	53	23
M_3	15	17	41	30
M_4	26	28	36	29

Step 1: Reduce each column by the largest figure in that column and ignore the resulting minus sign. The result is shown in the table below.

	W	X	Y	Z
M_1	13	10	30	16
M_2	0	13	0	7
M_3	23	11	12	0
M_4	12	0	17	1

Step 2: Reduce each row by the smallest figure in that row. The result is shown in the table below.

	W	X	Y	Z
M_1	3	0	20	6
M_2	0	13	0	7
M_3	23	11	12	0
M_4	12	0	17	1

Step 3: Cover zeros by the minimum possible number of lines

	W	X	Y	Z
M_1	3	0	20	6
M_2	0	13	0	7
M_3	23	11	12	0
M_4	12	0	17	1

Step 4: If the number of lines equals the number of assignments to be made go to step 6. If less as in the current example , select the smallest element which is not covered by the lines. In the current example, it is 3. Subtract this number from all the elements which are not covered. Then add this number to all those covered elements that are at the intersection of two lines. This gives the following reduced cost matrix

	W	X	Y	Z
M_1	0	0	17	6
M_2	0	16	0	10
M_3	20	11	9	0
M_4	9	0	14	1

Step 5: A feasible assignment is now possible and an optimal solution is assigned.

$M_1 \rightarrow$ Z

$M_2 \rightarrow$X

$M_3 \rightarrow$ W

$M_4 \rightarrow$ Y

Step 6: Calculate the contribution to be gained from the assignments.

$M_1 \rightarrow$ Z$\rightarrow$ \$30

$M_2 \rightarrow$X$\rightarrow$\$28

$M_3 \rightarrow$ W$\rightarrow$\$25

$M_4 \rightarrow$ Y$\underline{\rightarrow \$53}$

Total= $\underline{\underline{\$136}}$

SELF ASSESSMENT QUESTIONS

QUESTION 1

A company employs service engineers based at various locations throughout the country to service and repair their equipment installed in customers' premises. Four requests for service have been received and the company finds that four engineers are available. The distances each of the engineers is from the various customers is given in the table below.

	J_1	J_2	J_3	J_4
M_1	25	18	23	14
M_2	38	15	53	23
M_3	15	17	41	30
M_4	26	28	36	29

The company wishes to assign engineers to customers to minimize the total distance to be traveled. Show how this could be done.

QUESTION 2

A foreman has four fitters and has been asked to deal with five jobs. The times for each job are estimated as follows.

FITTERS

JOB	F_1	F_2	F_3	F_4
1	6	12	20	12
2	22	18	15	20
3	12	16	18	15
4	16	8	12	20
5	18	14	10	17

Allocate the men to the jobs so as to minimize the total time taken. Identify the job, which will not be dealt with.

QUESTION 3

A company has four salesmen who have to visit four clients. The profit records from previous visits are shown in the table below.

	SALESMEN			
CUSTOMERS	A	B	C	D
1	$6	$12	$20	$12
2	$22	$18	$15	$20
3	$12	$16	$18	$15
4	$16	$8	$12	$20

Show how the salesmen would be assigned to the clients so as to maximize profit.

QUESTION 4

A group of four boys and girls are planning on a day picnic. The extent of mutual happiness between boy i and girl j when they are together is given in the matrix below (data obtained from their previous dating experience).

	GIRL			
BOY	1	2	3	4
1	11	1	5	8
2	9	9	8	1
3	10	3	5	10
4	1	13	12	11

The problem is to decide the proper matching between the boys and the girls during the picnic, which will maximize the sum of all the mutual happiness of all the couples. Formulate this as an assignment problem and solve.

QUESTION 5

A batch of four jobs can be assigned to five different machines. The set-up time for each job on various machines is given by the following table.

MACHINE

JOB	1	2	3	4	5
1	10	11	4	2	8
4	7	11	10	14	12
3	5	6	9	12	14
4	13	15	11	10	7

Find the optimal assignment of jobs to machines, which will minimize the total set-up time

UNIT 4

NETWORK ANALYSIS

OUTLINE

OBJECTIVES

By the end of this unit you should be able to:

3. Distinguish between AOA and AON notations
4. Draw network diagrams using AON and AOA notations.
5. Calculate normal project completion time and cost
6. Calculate and explain floats, early and late times of activities
7. Explain crashing and calculate crash cost of an activity
8. Distinguish between CPM and PERT

Note: In order to achieve these objectives, you need to spend a minimum of four (4) hours and a maximum of six (6) hours working through the sessions.

SESSION 4.1: INTRODUCTION

This is a tool for organizing and planning large-scale projects, which consists of a list of smaller tasks that need to be completed at various stages in order for the project to progress and to be completed on time. Network analysis is a generic term for a family of related techniques developed to aid management to plan and control projects. These techniques show the interrelationship of the various jobs or tasks, which make up the overall project and clearly identify the critical parts of the project. They can provide planning and control information on the time, cost and resource aspects of a project.

Network analysis is likely to be of most value where projects are:

1. Complex. That is, they contain many related and interdependent activities.
2. Large, that is many types of facilities (high capital investment, many personnel, etc) are involved.
3. Where restrictions exists. That is where projects have to be completed within stipulated time or cost limits, or where some or all of the resources are limited.

Such projects might be for example:

1. The construction of a transport link;
2. Preparing a dinner party (organizing a wedding)
3. Launching a new product. Etc.

Each of these projects involves a large number of activities, which all need to be completed in a certain order before the whole project is finished. Some activities can be performed simultaneously, whilst others can only be tackled once other activities have finished. For example if you are planning a building, clearly the plastering of the internal walls can be undertaken at the same time as the grounds or landscaped. However, the walls cannot be built until the foundations are laid.

Here we can see a natural precedence relation between the activities and as each activity can only be completed in a certain time scale, the project manager faces the dilemma of carrying out all the activities in the best time and at reasonable cost so that the project completes on time and meets the budget constraints.

Businesses are constantly working on projects and from your own experience, you will know that management of these projects is very important. There are a number of methods that can be used to successfully manage a project, but we will be focusing on one particular aspect - time: extremely important to all businesses we acknowledge.

An unexpected delay or similar time crisis can put a business under a lot of pressure. So, how can you plan your time and identify where delays could occur, leaving you to make decisions to compensate for such situations? A useful tool to use would be the Critical Path Analysis(CPA**)**

SESSION 4.2: CRITICAL PATH ANALYSIS

Critical Path Analysis can be defined as the logical sequencing of a series of events necessary for a successful research project in such a manner that the most efficient route to some culmination point can be calculated. Consequently, the critical path technique has a multitude of uses: (a) As an aid in time management, (b) as a provider of ongoing data for assessing progress, and (c) to give the researcher or program planner considerable information for decision-making.

The critical path analysis is a tool that illustrates the individual tasks of a project highlighting the expected starting and finishing tasks of each. More precisely, the critical path analysis can be used to:

- Estimate the minimum/maximum time that tasks will be started and completed
- Estimate the minimum time that the whole project will take to complete
- Identify if resources are not being used effectively
- Make aware any tasks that could create a possible delay
- Sequence logically of activities that must be made

Ultimately, the critical path analysis will suggest which tasks are critical to keep on time anticipating that the delay in any one of the tasks will delay the whole project.

The critical path analysis is hard to explain in more detail without the use of diagrams, and so the working example in the following section will make it all clearer.

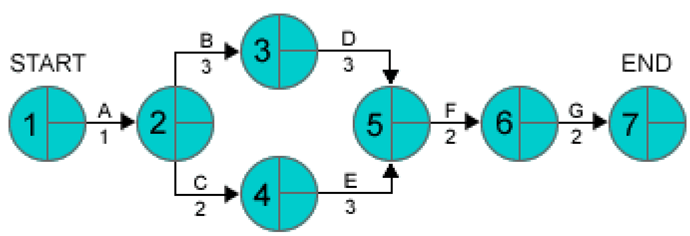
Illustration

Jubilant Limited have decided to carry out some research to ultimately create a selling strategy for their new product. They have decided to create a questionnaire, which they will issue to the public personally. In addition, they will use a mail shot to send out a similar survey to get the opinions of those that live outside of the area.

Each task (A - F) has been given an expected completion time (in weeks). Time is crucial for Jubilant Limited and so the Manager has requested a Critical Path Analysis of the project

Task	Description	Order/Logic	Time
A	Plan Primary Research	To be completed first	1 wks
B	Prepare Mail Shot (Postal Survey)	Start when A is complete	3 wks
C	Prepare Questionnaire	Start when A is complete	2 wks
D	Send and Wait for Mail Shot Replies	Start when B is complete	3 wks
E	Issue Questionnaire	Start when C is complete	3 wks
F	Compile and Analyze Results	Start when D & E is complete	2 wks
G	Plan Selling Campaign	Start when D, E & F is complete	2 wks

Now, take a good look at the diagram below. Each circle (Node) will be used to enter specific data. The numbers currently in the nodes (1 - 7) are only there to make following the diagram easier - nothing more. The arrows represent the tasks and each is given their respective completion times.



Before we move on, look at the above table again and ensure that you understand what is being said by linking it back to the diagram. Basically, all tasks cannot start until the previous task has been completed. This is not true for tasks B and C which can start at the same time - this is the only tricky area.

Each of these projects involves a large number of activities, which all need to completed in a certain order before the whole project is finished. Some activities can be performed simultaneously, whilst others can only be tackled once other activities have finished. For example if you are planning a building, clearly the plastering of the internal walls can be undertaken at the same time as the grounds or landscaped. However, the walls cannot be built until the foundations are laid. Here we can see a natural precedence relation between the activities, and as each activity can only be completed in a certain time scale, the project manager faces the dilemma of carrying out all the activities in the best time and at reasonable cost so that the project completes on time and meets the budget constraints.

For each activity, there is a set of activities called predecessors of the activity that must be completed before the activity begins. We need a list of all the activities that make up the project. The project is considered complete when all the activities have been completed

4.1.1: CPM Limitations

CPM was developed for complex but fairly routine projects with minimal uncertainty in the project completion times. For less routine projects there is more uncertainty in the completion times, and this uncertainty limits the usefulness of the deterministic CPM model. An alternative to **CPM is the PERT** project planning model, which allows a range of durations to be specified for each activity. CPM is a deterministic method that uses a fixed time estimate for each activity. While CPM is easy to understand and use, it does not consider the time variations that can have a great impact on the completion time of a complex project.

SESSION 4.3: TERMINOLOGIES

1. Activity: This is a task or job of work, which takes or consumes time and resources. For example, “Build a wall”, Dig foundations for a building”, “Verify the names of debtors in a sales ledger’, etc.

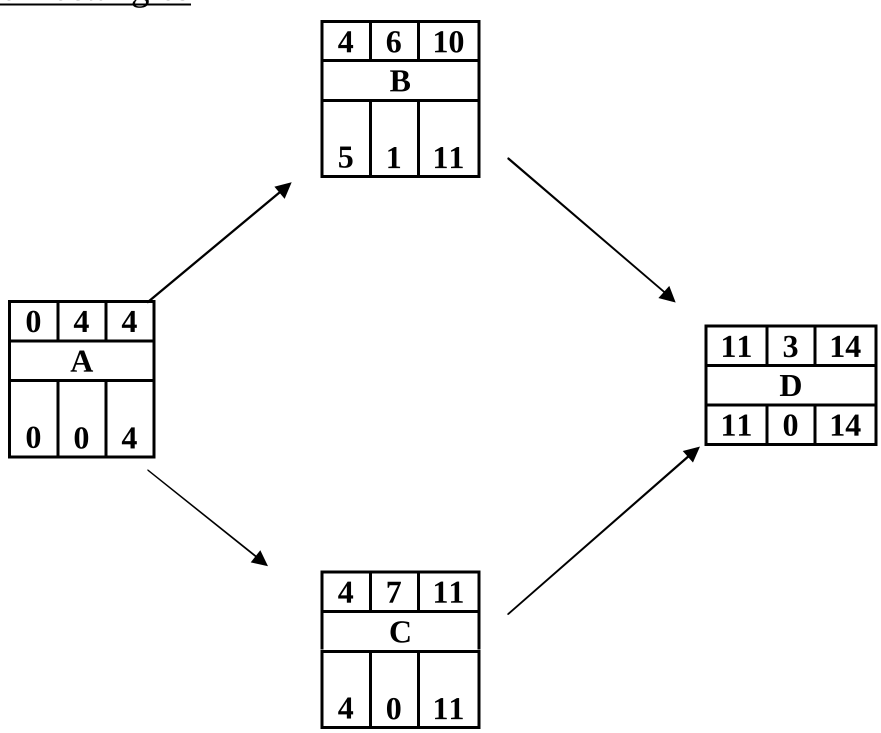
An activity is represented in a network by an arrow as shown . The tail of the arrow indicates where the task begins, and the head where the task ends. The arrow points from left to right and it is not drawn to scale.

2. Event: This is a point in time and indicates the “start” or “finish” of an activity or activities. An event is represented in a network by a circle or node as shown

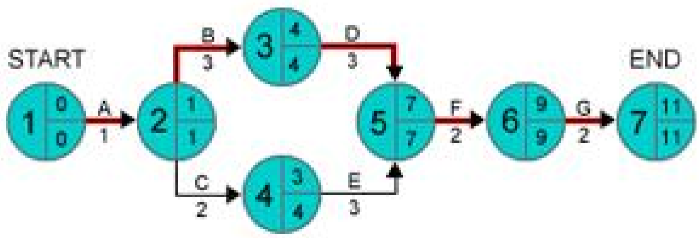
3. Dummy activity: This is an activity, which does not consume time or resources. It is used to merely show clear, logical dependencies between activities so as not to violets the rules for drawing networks. It is represented in a network by dotted arrow thus.

4. Network: This is the combination of activities, dummy activities and events in logical sequence according to the rules for drawing networks. Thus a small network might appear as shown below.

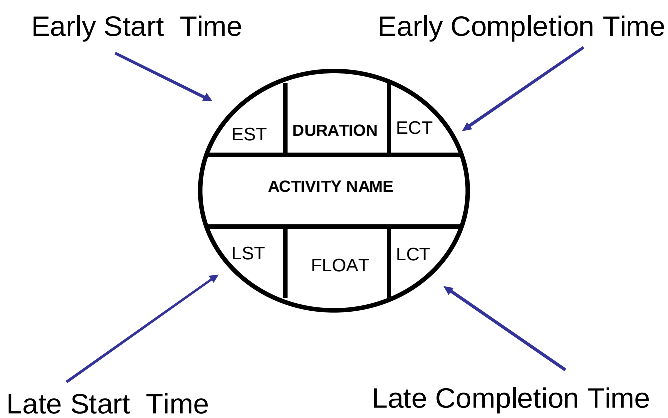
<u>Use of rectangles</u>



Use of circles



Note: The circles/rectangles are divided into sectors as



shown.

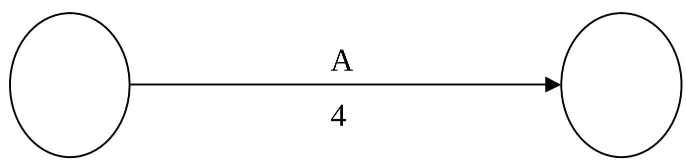
Early Start Time (EST)	Duration	Early Completion Time (ECT)
Activity Description		
Latest Start Time (LST)	Float	Latest Completion Time (LCT)

Note: In a project, an activity is a task that must be performed and an event is a milestone marking the completion of one or more activities. Before an activity can begin, all of its predecessor activities must be completed. Project network models represent activities and milestones by arcs and nodes.

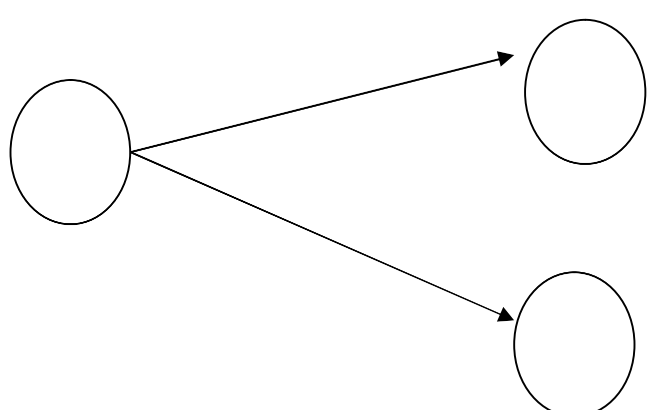
SESSION 4.4: NOTATIONS FOR DRAWING NETWORK DIAGRAMS

A network diagram is a graphical representation of the entire project. Each activity in the project is represented by a circle (node) or rectangle. Arrows are used to indicate sequencing requirements.

There are two main notations used in network diagrams. The first is arrow diagram notation where each activity is represented by an arrow (or a line) joining two circles (or nodes). The nodes represent transitions between activities, which are referred to as events. The duration of the activity is written against the arrow representing the activity. This type of representation is called Activity on Arc (AOA). The diagram below shows activity A in an arrow diagram. The duration of A being 4 days. The nodes are numbered beginning from the “start” node and then move from left to right.



The second notation is the Activity on Node (AON) diagram notation. Here each activity is represented by a circle (or node) and the arcs(lines) represent the events which are the transitions between activities. The duration of the activity is written against the arrow emerging from its node (circle). The diagram below shows activity A in AON notation, which has two successors, B and C. The name of the activity is written in the node(circle) in this case.



The following rules are used when drawing network diagrams**:**

i. Node 1 represents the ‘start” of the project. An arc or arcs should lead from node 1 to represent such activity that has no predecessor. A network has only one “start” node.

ii. A node (called the “finish” node) representing the completion of the project should be included in the network. A network has only one “finish” node.

iii. Number the nodes in the network so that the node representing the completion of an activity always has a larger number than the node representing the beginning of an activity. (There may be more than one numbering scheme)

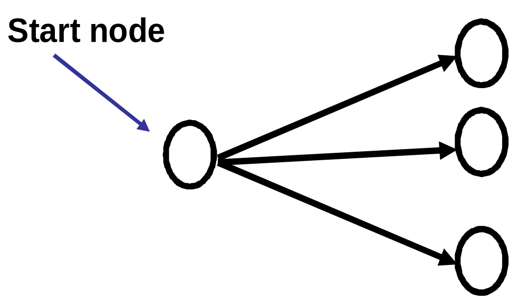
iv. An activity should not be represented by more than one arc in the network

v. Two nodes can be connected by at most one arc. That is activities should not share the same “start” node (tail event) and “finish” node (head event).

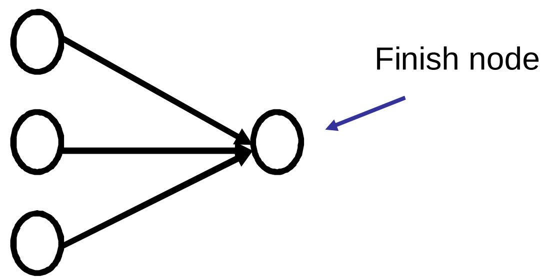
vi. Every activity must have one preceding event (tail event) and one succeeding event(head event).

Note that

1. Many activities may use the same “start” node (tail event).



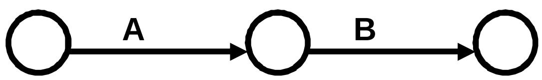
2. Many activities may use the same "finish" node (head event)



Illustrations - 1

(1). Suppose activity A is a predecessor of activity B, then we have:

AOA

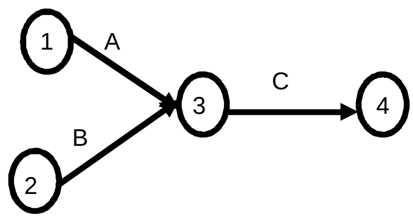


AON

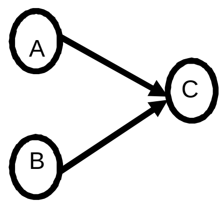
Illustrations - 2

2. Suppose activity A and B must be completed before activity C can begin and that A and B do not start at the same time but are performed simultaneously

AOA



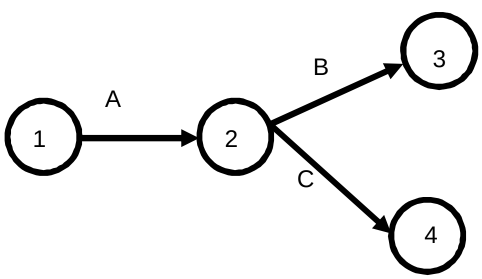
AON



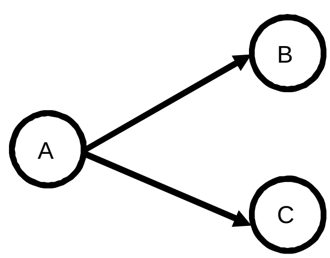
Illustrations - 3

3. Suppose activity A is a predecessor of activities B and C.
(B and C start from the same 'start' node).

AOA



AON



SESSION 4.5: CONSTRUCTING NETWORK DIAGRAMS

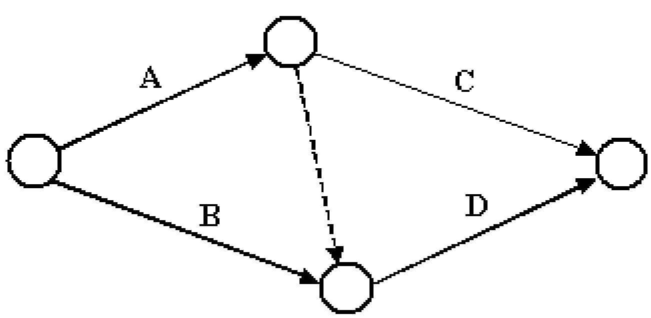
The first stage in constructing a network diagram for a project is to consult with the technical experts and summarise the information that they provide in a table, as in the following simple example.

Activity	Predecessors	Duration
A	None	4 months
B	None	3 months
C	A	2 months
D	A, B	4 months

The **predecessors** impose constraints on the project: C cannot begin until A has been completed, whereas D cannot begin until both A and B have been completed.

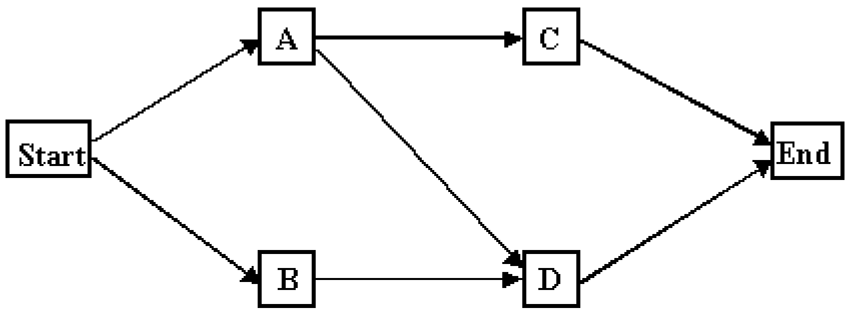
There are two types of network that we can draw for this project:

1. **Activity-on-arrow** (AOA)



The **Dummy Activity** (shown by the dotted line) is needed in the AOA network to show the predecessors correctly, i.e. that D depends on both A and B, but C depends only on A. AOA networks often need several dummy activities. AON networks do not need dummy activities.

2. **Activity-on-node** (AON)



Which type of network is better? Students often find AON networks a little easier to draw, and this can seem like a big advantage when doing it for the first time. However, some of the more advanced types of network analysis are best done using an AOA network. Every well educated student should be able to draw and analyse either, so that they are never thrown into confusion if they come across the one they haven't learnt to use. For people in employment, if your boss has a preference, then that's the one to use.

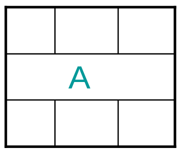
SESSION 4.6: ACTIVITY ON NODE (AON) DIADRAMS

Data below will be used to illustrate the drawing of AON diagram

Example 1: Draw a network diagram using the following data.

Activity	Preceding activity	Duration
A	-	4
B	A	6
C	A	7
D	B, C	3

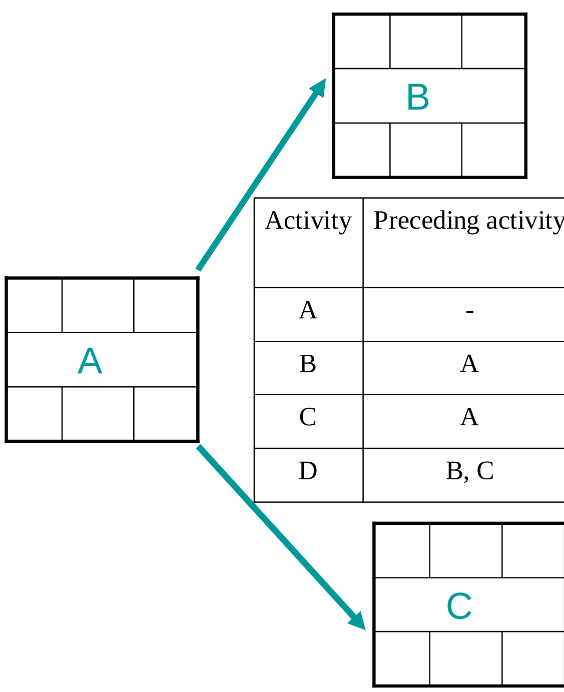
Step 1: Since activity A does not have any predecessor, it uses the first node as shown



Activity	Preceding activity	Duration
A	-	4
B	A	6
C	A	7
D	B, C	3

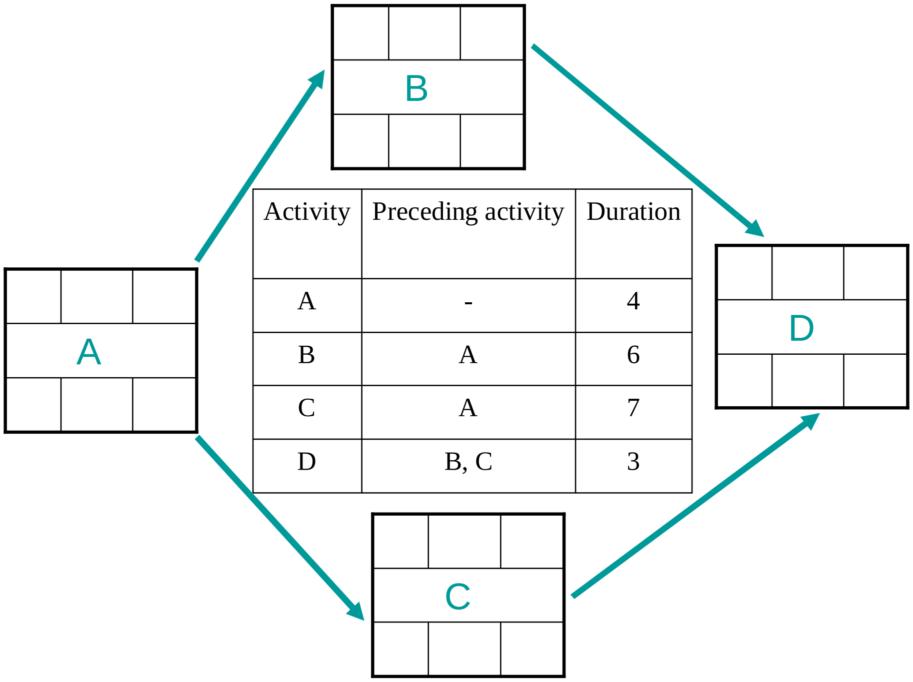
Note 1: Node 1 represents the 'start" of the project. An arc or arcs should lead from node 1 to represent such activity that has no predecessor. A network has only one "start" node.

Step 2: From activity A, move to activities B, and C



Activity	Preceding activity	Duration
A	-	4
B	A	6
C	A	7
D	B, C	3

Step 3: From activities B and C, move to activity D



Activity	Preceding activity	Duration
A	-	4
B	A	6
C	A	7
D	B, C	3

SESSION 4.7: CRITICAL PATH

The critical path of a network gives the shortest time in which the whole project can be completed. It is the chain of activities with the longest duration times.

Certain activities are critical to the on-line completion of the project. For example, if you are planning a dinner party and spend all your time setting the table and forget to put the main dish in the oven, the dinner will be late. This also means that other activities have float or slack time (time to spare) available such as laying the table

From the network diagram, the paths are:

1. A through C to D (4 + 6 + 3 = 13)
2. A through B to D (4 + 7 + 3 = 14)

The longer of this is A through C to D. Therefore, the critical path is: A through C to D

SESSION 4.8: EARLY TIMES

The Early Start Time (EST) is the earliest time at which the event corresponding to node i can occur Calculations begin from the 'start' node and move to the 'finish' node. At each node a number is computed representing the Earliest Start Time (EST).

Note:

1. Let EST_i be the earliest start time of event i. If i=1 is the start event time of event i, then $EST_i = 0$. That is, the EST of the first activity in any network diagram is zero (0)

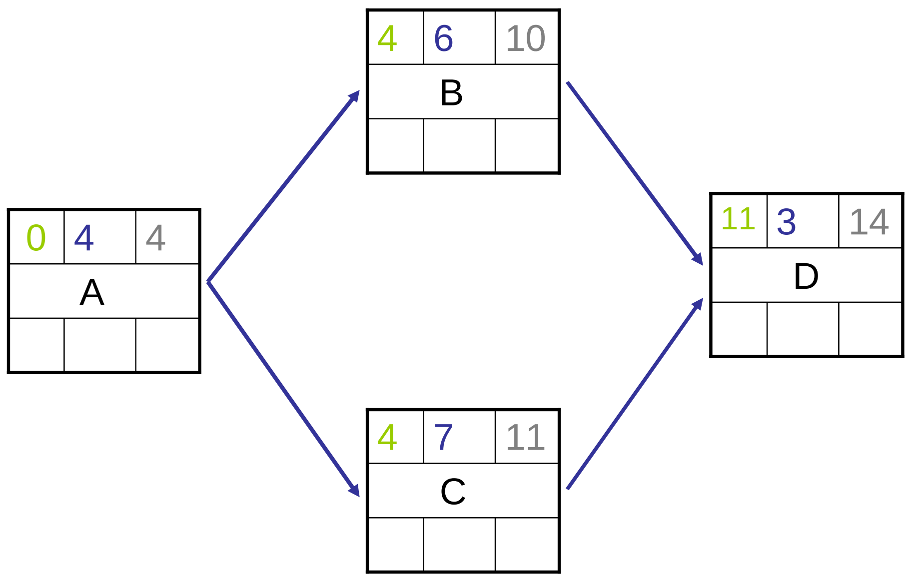
2. Let D_i be the duration of activity i. Then the Early Times calculations are obtained from the formula: $EST_j = Max(EST_i)$
That is the EST of any activity is equal to the maximum of the ECT of the preceding activities.

3. $ECT_i = EST_i + D_i$. That is the ECT of any activity is equal to its EST plus its duration.

4. The process of computing the Early Times is called Forward Pass.

The computations are shown in the diagram below.

EARLY TIMES



SESSION 4.9: LATE TIMES

The Late Completion Time (LCT) for node i represented by LCT_i is the latest time at which the event corresponding to node i can occur without delaying the completing of the project. Calculations begin from the 'finish' node and move to the 'start' node. At each node a number is computed representing the Latest Completion Time (LCT) the corresponding event.

<u>Note:</u>

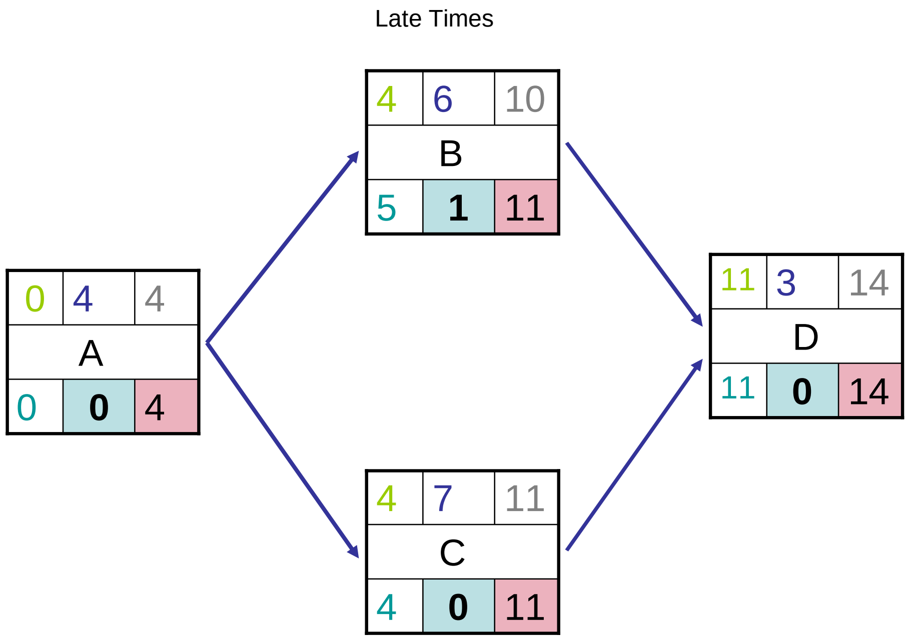
1. $LCT_{FinishNode} = ECT_{FinishNode}$. That is LCT of the last activity in a network diagram is equal to the ECT of the last activity.

2. $LCT_i = Min(LST_j)$. That is the smallest of the earliest start times for all activities that immediately follow the activity.

3. $LST_i = LST_i - D_i$. That is the LST of any activity is equal to its LCT minus its duration.

4. The process of computing the Late Times is called <u>Back Pass.</u>

The computations are shown in the diagram below.



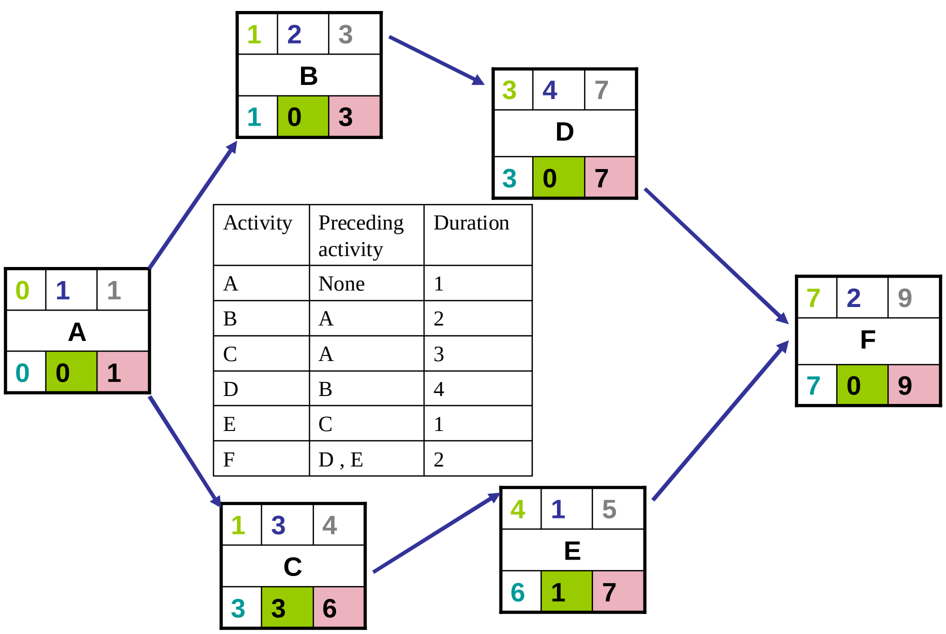
SESSION 4.10: FLOAT

Float (spare time or slack time) is the amount of time a path of activities could be delayed without delaying the overall project. A float can only be associated with activities, which are non-critical. By definition, activities on the critical path cannot have a float (spare time). In other words, the float for an activity is the difference between the maximum time available for that activity and the duration of that activity. The float for an activity with given by:

Float = Latest Start Time – Earliest Start Time (F= LST– EST) or (LCT-ECT)

Example 1

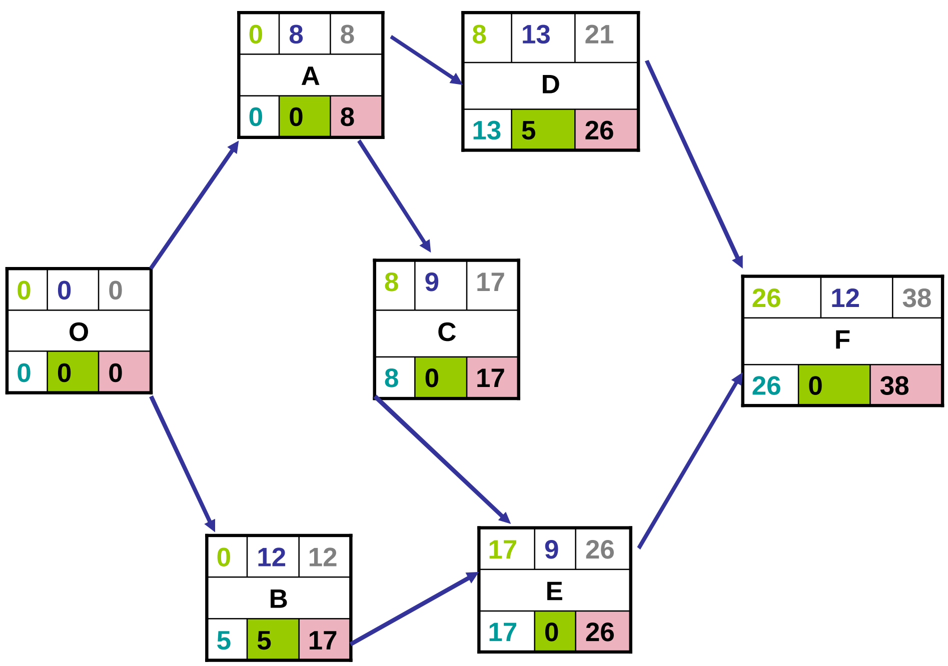
Activity	Preceding activity	Duration
A	None	1
B	A	2
C	A	3
D	B	4
E	C	1
F	D , E	2



EXAMPLE 2

Activity	Preceding activity	Duration
A	-	8
B	-	12
C	A	9
D	A	13
E	B , C	9
F	D , E	12

Solution



SESSION 4.11: PROJECT TIME REDUCTION

One important aspect of project management is looking at ways of reducing the overall project time at an acceptable cost. Reducing or shortening the duration of an activity is known as crushing in the CPM terminology. The normal duration of a project incurs a given cost and by employing more labour, working overtime, more equipment, etc, the duration could be reduced but at the expense of higher costs. Since some ways of reducing the project duration will be cheaper than others, network time/cost analysis seeks to find the cheapest way of reducing the overall duration. A common feature of many projects is a penalty clause for delayed completion and or bonus for earlier completion. Thus, project managers take these into consideration whenever undertaking any project.

We use a network technique known as Critical Path Method (CPM) to enable us to look at the order of activities in a project and to plan the time involved. CPM can be thought of as an application of graph theory. The main objectives of CPM are:

i. To locate the critical activities.
ii. To allocate the time of the other activities to obtain the most efficient use of labour and resources.
iii. To consider ways of reducing the total project time by speeding up the activities on the critical path while monitoring the overall project to ensure that the critical activities remain critical.

There may be more than one critical path in a network and it is possible for the critical path to run through a dummy activity. The critical calculation involves two phases:

1. **Crashing:** Reducing the duration of a project/activity at a minimum cost. Before we proceed, let us look at the following terminologies.

2. Normal Cost: The cost associated with a normal time estimate for an activity. Often, the normal time estimate is set at the point where resources are used in the most efficient manner.

3. Crash Cost: The cost associated with the minimum possible time for an activity. Crash costs, because of extra wages, overtime premiums, extra facility costs, are always higher than the normal cost.

4. Crash Duration/Time: The minimum time that an activity is planned to take. The minimum time is invariably brought about by the application of extra resources, for example, more labour or machinery.

5. Cost Slope: This is the average cost of shortening an activity by one time unit(day, week, month as appropriate). The cost slope is generally assumed to be linear and is calculated as follows:

$$\text{Cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}} = \frac{\text{Increase in cost}}{\text{Decrease in time}}$$

Illustration 1: Activity A has the following information:

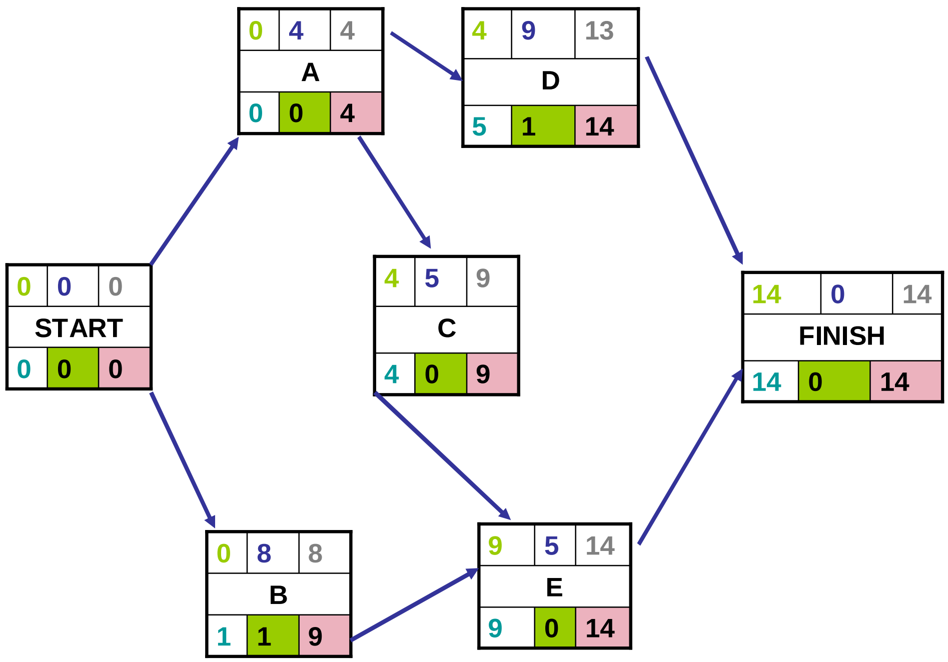
Normal time and cost: 12 days @ $480

Crash time and cost: 8 days @ $640

$$\text{Cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}} = \frac{\text{Increase in cost}}{\text{Decrease in time}} = \frac{640 - 480}{12 - 8} = \$40/\text{day}$$

Illustration 2: The table below shows the data for a simple project

Activity	Preceding activity	Normal time	Crash time	Normal cost	Crash cost	Cost slope
A	None	4	3	360	460	60
B	None	8	5	300	510	70
C	A	5	3	170	270	50
D	A	9	7	220	300	40
E	B , C	5	3	200	360	80



- Normal project duration = critical path: 1 →2 → 3 → 4 (i.e. activities A, C, and E) = 4+5+5=14 days.

OR the LCT of the Finish node can be use: LCT = !4(see network diagram)

- Normal Project Cost =Cost of all activities at normal time =360+300+170+220+200 =$1250

Steps for crashing

1. Draw the network diagram for the project.
2. Determine the critical activities.
3. Calculate the cost slopes for the critical activities.

Note:

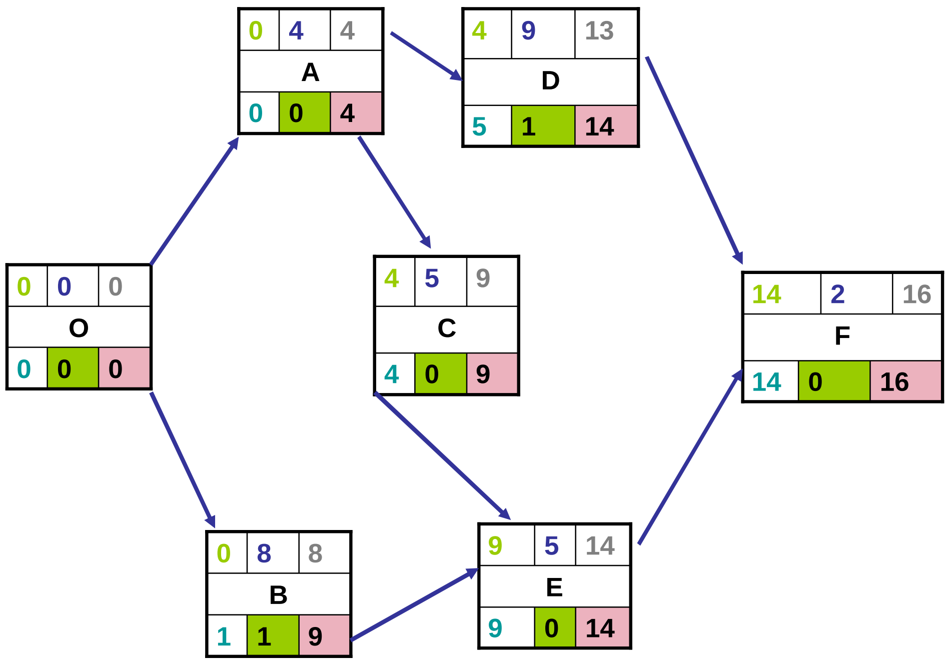
1. Only critical activities are crashed in order to reduce the project duration.
2. An activity cannot be crashed beyond its crash duration

Illustration.

Crashing:-Example 1

Activity	Pre activity	Normal time	Crash time	Normal cost	Crash cost
A	**None**	**4**	**3**	**360**	**460**
B	**None**	**8**	**5**	**300**	**510**
C	**A**	**5**	**3**	**170**	**270**
D	**A**	**9**	**7**	**220**	**300**
E	**B , C**	**5**	**3**	**200**	**360**
F	**D, E**	**2**	**1**	**200**	**250**

Step 1: Draw the network diagram



Step 2: Determine the critical activities. These are A, C, E and F

Step 3: Calculate the cost slopes for the critical activities

Activity	Pre activity	Normal time	Crash time	Normal cost	Crash cost	Cost slope
A**	None	4	3	360	460	100
B	None	8	5	300	510	
C**	A	5	3	170	270	50
D	A	9	7	220	300	
E**	B , C	5	3	200	360	80
F**	D, E	2	1	200	250	50

Now suppose we want to crash the project by 5 days(that is finish the project in 11 days instead of the 16 days), what will be the cost of the project within this time?

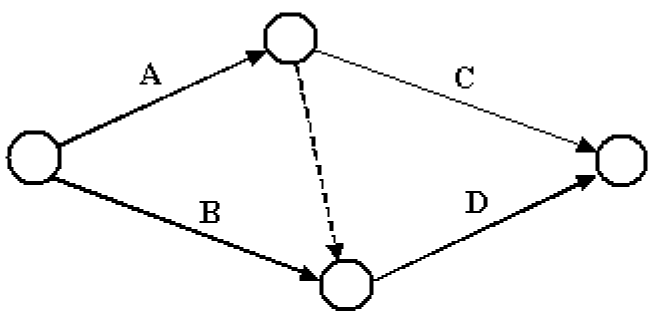
1. The normal project completion time is 16 days(LCT = 16)
2. The cost of completing the project in 16 days is $1450.00 (360+300+170+220+200+200)
3. Crashing is done as follows;
   i. Crash activity C by 2 days = $50 *2= $100.00
   ii. Crash activity F by 1 day = $50*1 = $50.00
   iii. Crash activity E by 2 days = $80*2=$160.00
   iv. Total cost of crashing =$310.00(100+50+160)
   v. Total cost of completing the project in 11 days =$1780.00(Normal cost of project + cost of crashing by 5 days). That is 1450 + 310

SESSION 4.12: ACTIVITY ON ARC (AOA) DIAGRAMS

The table will be used to demonstrate the drawing of AOA diagrams

Activity	Predecessors	Duration
A	None	4 months
B	None	3 months
C	A	2 months
D	A, B	4 months

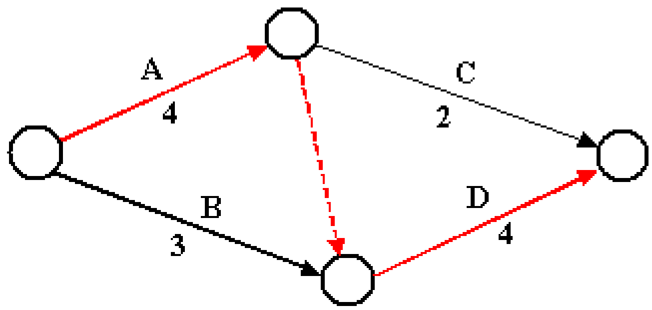
The network diagram for the above table is shown below:



The dummy activity (shown by the dotted line) is needed in the AOA network to show the predecessors correctly, i.e. that D depends on both A and B, but C depends only on A. AOA networks often need several dummy activities. AON networks do not need dummy activities.

1. Critical Activities

An activity is said to be critical if its duration cannot be extended without delaying the completion of the project.



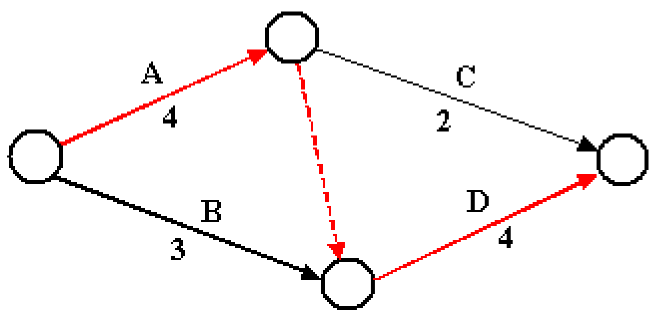
Notice that there are three paths through the network:

Path	A-C	B-D	A-D
Length	6	7	8

The duration of the project is determined by the longest path - D cannot begin until A has been completed. The critical activities are those which lie on a longest path through the network - here A and D are critical. Any extension or delay in a critical activity will make such a path longer still, thereby delaying completion of the project. A longest path through a network is called a critical path. On the diagram the critical path is shown in red.

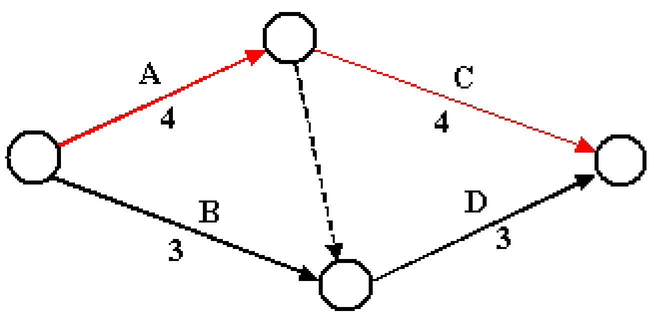
2. Floats

In the simple network below the critical path is shown in red.



Any activity that is not a critical activity can be extended or delayed by a certain amount without delaying the completion of the project. For example, activity C should start after 4 months and be finished after 6 months. However, the finish could be delayed by 2 months, i.e. until the end of month 8, without affecting the completion time of the project as a whole. We say that activity C has a total float of 2 weeks.The Total Float of an activity is the amount of time by which it may be extended or delayed without delaying completion of the project, assuming no extension or delay in any other activity. A critical activity can be defined as one which has a total float of zero.

There are two other types of float that are sometimes used. To understand these, suppose the durations are changed in the example above. The critical path is now A-C of duration 8 months, B now has a total float of 2 and D has a total float of 1.



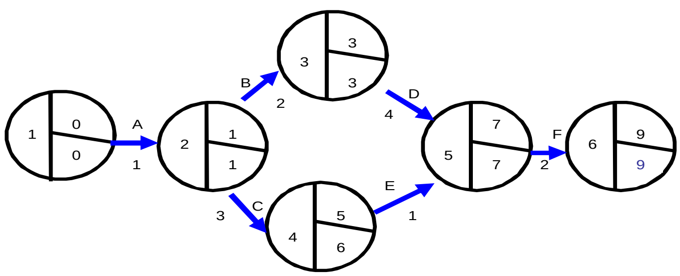
The Free Float of an activity takes into consideration the effect on subsequent activities of a delay. If B is delayed by 2 months it removes the spare time that was available for D, since if D was now also delayed by 1 the total project duration would extend to 9 months. However, B can be delayed by 1 month without reducing the spare time for any subsequent activity. Therefore, B has a free float of 1 month. D also has a free float of The Independent Float of an activity is calculated assuming the worst circumstances, i.e. the activity's predecessors finish at their latest times and we want subsequent activities to begin at their earliest times. If this is possible, and there is still time to spare, then this is called independent float. The free float of 1 month for B is also an independent float. However, assuming the worst scenario in which B is delayed by 2 months then D cannot be delayed at all. Hence D has an independent float of 0.

Example 2.

Draw a network diagram using AOA

Activity	Preceding activity	Duration
A	None	1
B	A	2
C	A	3
D	B	4
E	C	1
F	D , E	2

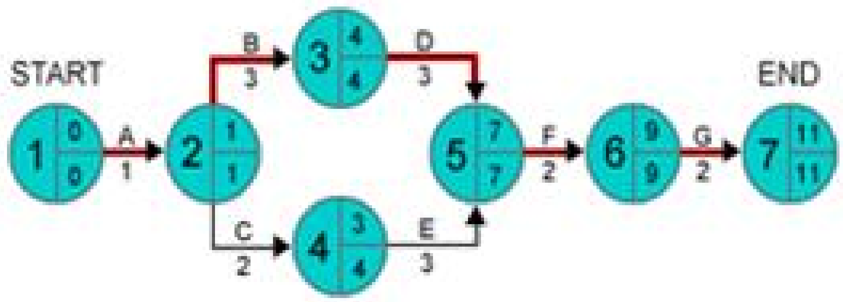
Network diagram-AOA



EXAMPLE 2

Task	Description	Order/Logic	Time
A	Plan Primary Research	To be completed first	1 wks
B	Prepare Mail Shot (Postal Survey)	Start when A is complete	3 wks
C	Prepare Questionnaire	Start when A is complete	2 wks
D	Send and Wait for Mail Shot Replies	Start when B is complete	3 wks
E	Issue Questionnaire	Start when C is complete	3 wks
F	Compile and Analyze Results	Start when D & E is complete	2 wks
G	Plan Selling Campaign	Start when D, E & F is complete	2 wks

EXAMPLE 3



SESSION 4.13: PROGRAM EVALUATION AND REVIEW TECHNIQUE(PERT)

The Program Evaluation and Review Technique commonly abbreviated PERT is a model for project management invented by United States Department of Defense's US Navy Special Projects Office in 1958 as part of the Polaris mobile submarine-launched ballistic missile project. This project was a direct response to the Sputnik crisis.

PERT is basically a method for analyzing the tasks involved in completing a given project, especially the time needed to complete each task, and identifying the minimum time needed to complete the total project. It was able to incorporate uncertainty in the sense that it was possible to schedule a project not knowing precisely the details and durations of all the activities. It is more of an event-oriented technique rather than start- and completion-oriented. This technique is used more in R&D-type projects where Cost is not a major factor but Time is.

The Program Evaluation and Review Technique (PERT) is a network model that allows for randomness in activity completion times. It has the potential to reduce both the time and cost required to complete a project.

PERT originally was an *activity on arc* network, in which the activities are represented on the lines and milestones on the nodes. Over time, some people began to use PERT as an *activity on node* network. For this discussion, we will use the original form of activity on node.

A distinguishing feature of PERT is its ability to deal with uncertainty in activity completion times. For each activity, the model usually includes three time estimates:

- *Optimistic time* - generally the shortest time in which the activity can be completed. It is common practice to specify optimistic times to be three standard deviations from the mean so that there is approximately a 1% chance that the activity will be completed within the optimistic time.

- *Most likely time* - the completion time having the highest probability. Note that this time is different from the *expected time*.
- *Pessimistic time* - the longest time that an activity might require. Three standard deviations from the mean is commonly used for the pessimistic time.

PERT assumes a beta probability distribution for the time estimates. For a beta distribution, the expected time for each activity can be approximated using the following weighted average:

Expected time = (Optimistic + 4 x Most likely + Pessimistic) / 6

This expected time may be displayed on the network diagram.

To calculate the variance for each activity completion time, if three standard deviation times were selected for the optimistic and pessimistic times, then there are six standard deviations between them, so the variance is given by:

$[(\text{Pessimistic} - \text{Optimistic}) / 6]^2$

4.13.1: Benefits of PERT

PERT is useful because it provides the following information:

- Expected project completion time.
- Probability of completion before a specified date.
- The critical path activities that directly impact the completion time.
- The activities that have slack time and that can lend resources to critical path activities.
- Activity start and end dates.

4.13.2: Limitations

The following are some of PERT's weaknesses:

- The activity time estimates are somewhat subjective and depend on judgement. In cases where there is little experience in performing an activity, the numbers may be only a guess. In other cases, if the person or group performing the activity estimates the time there may be bias in the estimate.
- Even if the activity times are well-estimated, PERT assumes a beta distribution for these time estimates, but the actual distribution may be different.
- Even if the beta distribution assumption holds, PERT assumes that the probability distribution of the project completion time is the same as the that of the critical path. Because other paths can become the critical path if their associated activities are delayed, PERT consistently underestimates the expected project completion time.

The underestimation of the project completion time due to alternate paths becoming critical is perhaps the most serious of these issues. To overcome this limitation, Monte Carlo simulations can be performed on the network to eliminate this optimistic bias in the expected project completion time.

4.13.3: Terminology

- A *PERT event*: is a point that marks the start or completion of one (or more) tasks. It consumes no time, and uses no resources.
- A PERT event that marks the completion of one (or more) tasks is not “reached” until all of the activities leading to that event have been completed.
- A *predecessor event*: an event (or events) that immediately precedes some other event without any other events intervening. It may be the consequence of more than one activity.

- A *successor event*: an event (or events) that immediately follows some other event without any other events intervening. It may be the consequence of more than one activity.
- A *PERT activity*: is the actual performance of a task. It consumes time, it requires resources (such as labour, materials, space, machinery), and it can be understood as representing the time, effort, and resources required to move from one event to another. A PERT activity cannot be completed until the event preceding it has occurred.
- *Optimistic time* (O): the minimum possible time required to accomplish a task, assuming everything proceeds better than is normally expected
- *Pessimistic time* (P): the maximum possible time required to accomplish a task, assuming everything goes wrong (but excluding major catastrophes).
- *Most likely time* (M): the best estimate of the time required to accomplish a task, assuming everything proceeds as normal.
- *Expected time* (T_E): the best estimate of the time required to accomplish a task, assuming everything proceeds as normal (the implication being that the expected time is the average time the task would require if the task were repeated on a number of occasions over an extended period of time).

  $T_E = (O + 4M + P) \div 6$

- *Critical Path*: the longest pathway taken from the initial event to the terminal event. It determines the total calendar time required for the project; and, therefore, any time delays along the critical path will delay the reaching of the terminal event by at least the same amount.
- *Lead time*: the time by which a *predecessor event* must be completed in order to allow sufficient time for the activities that must elapse before a specific PERT event is reached to be completed.
- *Lag time*: the earliest time by which a *successor event* can follow a specific PERT event.
- *Slack*: the slack of an event is a measure of the excess time and resources available in achieving this event. Positive slack would indicate *ahead of schedule*;

negative slack would indicate *behind schedule*; and zero slack would indicate *on schedule*

4.13.4: Steps In The PERT Planning Process

PERT planning involves the following steps:

1. Identify the specific activities and milestones.
2. Determine the proper sequence of the activities.
3. Construct a network diagram.
4. Estimate the time required for each activity.
5. Determine the *critical path*.
6. Update the PERT chart as the project progresses.

Illustrative Example

The following represents a project that should be scheduled using PERT

Activity	Predecessors	Time estimates		
		a	m	b
A	-	10	22	28
B	A	4	4	10
C	A	4	6	14
D	B	1	2	3
E	C,D	1	5	9
F	C,D	7	8	9
G	E,F	2	2	2

1. Calculate the expected time for each activity. This is given as follows: $E(t) = \frac{a + 4m + b}{6}$. Use the expected times to draw the network diagram
2. Determine the critical path. Using the expected times, a critical path is calculated in the same manner as in the single time estimate
3. Calculate the variance(σ^2) for the activity time: Specifically, this is the variance associated with each E(t) and is computed using the formula below: $\sigma^2 = \left(\frac{b - a}{6}\right)^2$
4. Determine the probability of completing the project on a given date. A valuable feature of using three time estimates is that it enables the analyst to assess the effect of uncertainty on project completion time. The mechanics of deriving the probability are as follows:

i. Sum the variance values associated with each activity.

ii. Substitute this figure along with the project due date and the project expected completion time into the transformation formula. This formula is: $z = \frac{D - E(t)}{\sqrt{\sum \sigma^2_{cc}}}$

Where:

D = Desired completion date for the project

E(t) = Expected completion date for the project

$\sum\sigma^2_{cc}$ = Sum of variances along the critical path.

iii. Calculate the value of Z, which is the number of standard deviations the project date is from the expected completion time.

iv. Using the value of Z, find the probability of meeting the project due date(using a table of normal probabilities)

Activity	Predecessors	Time estimates			Expected Time $E(t)=\frac{a+4m+b}{6}$	Activity Variance $\sigma^2=\left(\frac{b-a}{6}\right)^2$
		a	m	b		
A	-	10	22	28	21	9
B	A	4	4	10	5	1
C	A	4	6	14	7	$2\frac{2}{9}$
D	B	1	2	3	2	$\frac{1}{9}$
E	C,D	1	5	9	5	$1\frac{7}{9}$
F	C,D	7	8	9	8	$\frac{1}{9}$
G	E,F	2	2	2	2	0

Since there are two critical paths in the network(the above example), we must decide which variances to use in arriving at the probability of meeting the project due date. A conservative approach dictates using the path with the largest variance since this would focus management's attention on the activities most likely to exhibit broad variations. On this basis, the variances associated activities A, C, F, and G would be used to find the probability of completion. Thus:

$$\sum \sigma^2 = 9 + 2\frac{2}{9} + \frac{1}{9} + 0 = 11.89$$

Suppose management asks for the probability of completing the project in 35 weeks, then D = 35. The expected completion time was found to be 38. Substituting these values in Z, we have:

$$z = \frac{D - E(t)}{\sqrt{\sum \sigma^2{}_{cc}}} = \frac{35 - 38}{\sqrt{11.89}} = -0.87$$

Using a table of normal probabilities, we see that a Z value of –0.87 yields a probability of 0.19, which means that the project manager has only 19% chance of completing the project in 35 weeks.

SELF ASSESSMENT QUESTIONS

QUESTION 1

The table below shows data for a simple project.

ACTIVITY	PRECEDING ACTIVITY	DURATION	COST	CRASH DURATION	COST
A	-	5	100	4	200
B	-	4	120	2	160
C	A	10	400	4	1000
D	B	7	300	3	700
E	B	11	200	10	250
F	C	8	400	6	800
G	D, E, F	4	300	4	300

i. Draw a network diagram for the project above.

ii. Find the critical path.

iii. Find the normal duration time of the project.

iv. Find the cost associated with the normal duration time.

v. Find the cost associated with crashing for three. days

QUESTION 2

The table below shows data for a simple project.

Activity	Preceding activity	Normal duration	Normal cost	Crash duration	Crash cost
A	-	3	1200	2	1500
B	-	2	1250	1	1300
C	A	6	1000	4	1800
D	A	5	1500	3	2800
E	C, D	5	1800	4	3000
F	B ,D	3	500	1	1000
G	F	2	500	1	1000

i. Draw a Network diagram for the project using (a) Activity on arc notation (AOA) and (b) activity on node notation (AON)

ii. Find the critical path.

i. Find the normal duration time of the project.

ii. Find the cost associated with the normal duration time.

iii. Find the shortest possible completion time of the project.

vi. Find the cost associated with the shortest possible completion time

QUESTION 3

The table below gives data for a simple project

Activity	Preceding activity	Normal duration	Normal cost	Crash duration	Crash cost
A	-	4	25	3	37
B	A	4	16	4	16
C	-	5	35	2	75
D	A , C	8	49	7	95
E	B , C	8	60	4	130
F	D ,E	6	41	3	76
G	E ,F	5	23	3	45
H	G	4	40	3	65

i. Calculate and state the critical path(s) and the cost of completion in the normal time

ii. Calculate and state the critical path(s) and the cost of completion in the crash time

iii. Calculate and state the minimum cost of completion and associated critical path(s).

QUESTION 4

The table below gives data for a simple project

Activity	Preceding activity	Normal duration	Normal cost	Crash duration	Crash cost
A	-	8	100	6	200
B	-	4	150	2	350
C	A	2	50	1	90
D	A	10	100	5	400
E	B	5	100	1	200
F	C , E	3	80	1	100

i. Draw a Network diagram for the project using activity on node notation (AON)

ii. Find the critical path.

iii. Find the normal duration time of the project.

iv. Find the cost associated with the normal duration time.

v. Find the shortest possible completion time of the project.

vi. Find the cost associated with the shortest possible completion.

QUESTION 5

The Ressembler Group is looking at the possible test launch of a new type of picture frame called 'Dale'. The main activities have been identified. Times estimated in weeks and costs estimated in dollars are shown below.

.Activity	Preceding Activity	NORMAL		CRASH	
		Time (weeks)	Cost ($)	Time (weeks)	Cost ($)
A	-	4	10,000	2	11,000
B	A	3	6,000	2	9,000
C	A	2	4,000	1	6,000
D	B	5	14,000	3	18,000
E	B, C	1	9,000	1	9,000
F	C	3	7,000	2	8,000
G	E , F	4	13,000	2	25,000
H	D , E	4	11,000	1	18,000
I	H, G	6	20,000	5	29,000

(a)Draw the network for this project.(Use AOA notation)

i. State the critical path and critical activities

ii. Find the normal project time.

iii. Find the normal project cost.

(b). The project director decides to shorten the project duration by three (3) weeks.

i. What activities should be crushed?

ii. What will be the total cost of the project in this time?

QUESTION 6

A manufacturing concern has received a special order for a number of units of a special product that consists of two components parts: X and Y. The product is a nonstandard item that the firm has never produced before, and scheduling personnel have decided that the application of CPM is warranted. A team of manufacturing engineers has prepared the following table.

Activity	Description	Predecessors	Expected Time(days)
A	Plan production	-	5
B	Procure materials for part X	A	14
C	Manufacture part X	B	9
D	Procure materials for part Y	A	15
E	Manufacture part Y	D	10
F	Assemble parts X and Y	C, E	4
G	Inspect assemblies	F	2
H	Completed	G	0

i. Construct a graphic representation of the data above

ii. Identify the critical path

iii. What is the length of time to complete the project?

iv. Find the floats

v. Find the possible number of days by which the project duration can be reduced and state the activities that need to be reduced.

QUESTION 7

The following represents a project that should be scheduled using PERT

Activity	Immediate predecessor(s)	Time(Days)		
		a	m	b
A	-	1	3	5
B	A	1	2	3
C	A	1	2	7
D	A	2	3	4
E	B	3	4	11
F	C , D	3	4	5
G	D , E	1	4	6
H	F , G	2	4	5

a. Draw the network diagram

b. What is the critical path?

c. What is the expected project completion time?

d. What is the probability of completing the project within 16 days?

QUESTION 8

The table below represents a plan for a project

Job No.	Immediate predecessor(s)	Time(Days)		
		a	m	b
1	-	2	3	4
2	1	1	2	3
3	1	4	5	12
4	1	3	4	11
5	2	1	3	5
6	3	1	2	5
7	4	1	8	9
8	5 , 6	2	4	6
9	8	2	4	2
10	7	3	4	5
11	9 ,10	5	7	8

a. construct an appropriate network diagram

b. Indicate the critical path

c. What is the expected completion time for the project?

UNIT 5

INVENTORY CONTROL

OUTLINE

Session 5.1: Introduction to inventory control

Session 5.2: Types of Inventory

Session 5.3: Terminologies

Session 5.4: The Functions of Inventory

Session 5.5: Inventory Policies

Session 5.6: Inventory Models/Systems

Session 5.7: Economic Order Quantity (EOQ) model

OBJECTIVES

By the end of the unit you should be able to:

1. Explain the rationale for inventory control
2. identify types of inventory
3. Explain the inventory policies
4. Use the inventory models to calculate order quantity

SESSION 5.1: INTRODUCTION TO INVENTORY CONTROL

Inventory refers to idle goods or materials held by an organization for use sometime in the future. Items carried in inventory include raw materials, purchased parts, components subassemblies, work – in-process, fished goods, and supplies. Some reasons why organizations maintain inventory are the difficulties in predicting sales levels, production times, demand and usage needs exactly. Thus, inventory serves as a buffer against uncertain and fluctuating usage and keeps a supply of items available in case the items are needed by the organizations or its customers. Even though inventory serves an important and essential role, the expense associated with financing and maintaining inventories is a substantial part of the cost of doing business. In large organizations, the cost associated with inventory can run into millions of dollars.

Good inventory management is essential to the success of most organizations if not all. The reasons are that:

1. Inventory represents money. In other words money is tied up in inventory.
2. Inventory has a very great impact on the daily operations of organizations, whether manufacturing or trading organizations.

Manufacturing organizations need stock of raw materials that will be converted to finished goods and this should not be in short supply at any point in time. Trading organizations need inventory of finished goods to meet the demands of their customers.

SESSION 5.2: TYPES OF INVENTORY

To accommodate the functions of inventory, firms keep some types of inventories;

i. Raw Material Inventory: These are materials that are usually purchased but are yet to enter the manufacturing process. These items can be used to decouple (separate) supplies from the production process.

ii. Work in Progress Inventory: They are incomplete products or components that are no longer considered raw materials but are yet to become finished products.

iii. Maintenance/Repair/Operation supply Inventory: They are inventories devoted to the maintenance/repair/operating supplies to keep machinery and processes productive. They exist because the need and timing for maintenance and repair of some equipment are unknown.

iv. Finished goods Inventory: They are completed/finished products awaiting shipment. Finished goods may be inventoried because future customers, demands are unknown.

SESSION 5.3: TERMINOLOGIES

i. Ordering/Setup Cost: It represents the fixed charge incurred when an order is placed. It is also cost associated with producing a good internally.

ii. Holding/Carrying Cost: Is the cost of carrying inventory in stock. The holding cost usually includes storage cost, insurance cost, interest on invested capital, depreciation and maintenance, theft etc.

iii. Demand: The amount required by sales or production

iv. Re-order Level: The level of stock at which a further replenishment order should be placed. The re-order level is dependent upon the lead time and the demand.

v. Lead Time: The period of time, expressed in days, weeks, etc. between ordering and replenishment.

vi. Re-order Quantity: The quantity of the replenishment order.

vii. Safety Stock: The level of stock that is maintained to provide some level of protection against stockouts.

SESSION 5.4: FUNCTIONS OF INVENTORY

1. Provide a stock of goods to meet anticipated customer demand and provide a “selection” of goods
2. Decouple suppliers from production and production from distribution
3. Allow one to take advantage of quantity discounts
4. To provide a hedge against inflation
5. To protect against shortages due to delivery variation
6. To permit operations to continue smoothly with the use of “work-in-process”

SESSION 5.5: INVENTORY POLICIES

An organisation's stockholding policy is guided by a series of rules, which determine when certain decisions concerning the holding of stocks should be made. the rules help managers to answer two important questions:

1. How much should be ordered when the inventory is replenished?
2. When should the inventory be replenished?

This series of rules is known as Inventory Policy.

i. **Re-order Level Policy:** If the decision concerning replenishment is based on the level of inventory held, it is known as Re-order Level Policy. A predetermined re-order level is set for each item. When the stock level falls to the re-order level, a replenishment order is issued. Fixed quantities are ordered at variable intervals. The replenishment order quantity is invariably the Economic Order quantity. This policy m, has three control levels:

   a. Reorder level: It is a definite action level. It is calculated so that if the worst anticipated position occurs, stock would be replenished. This level is given as Maximum usage multiplied by Maximum lead time.

   b. Minimum level: This level is calculated so that management will be warned when demand is above average and accordingly buffer stock is being used. The Minimum level is given as Re-order level minus Average usage for average lead time.

   c. Maximum level: It is calculated so that management will be warned when demand is the minimum anticipated and consequently the stock level is likely to rise above the maximum intended. The Maximum level is given as Re-order level plus Economic Order quantity minus Minimum anticipated usage in lead time.

ii. **Periodic Review Policy:** This is also called the constant cycle system. The stock levels for all parts are reviewed at fixed intervals for example, every fortnight. The quantity of the replenishment order is not a previously calculated EOQ, but is based upon the likely demand until the next review, the present stock level and the lead time. Variable quantities are ordered at fixed intervals.

Example

The following data relate to a particular stock item.

Normal usage is 110 per day

Minimum usage is 50 per day

Maximum usage is 140 per day

Lead time is 25 to 30 days

Economic Order Quantity is 5,000

Using the data above, calculate the various control levels

Solution:

i. Re-order level = Maximum usage x Maximum lead time = 140 x 30 = 4,200 units

ii. Minimum level = Re-order level - Average usage for average lead time.

= 4,200 –(110 x 27.5) = 1,175 units.

iii. Maximum level = Re-order level + Economic Order quantity - Minimum anticipated usage in lead time = 4,200 + 5,000 – (50 x 25) = 7,950 units.

SESSION 5.6: INVENTORY MODELS/SYSTEMS

There are three inventory model that address important questions such as:

i. When to order?
ii. How much to order?

These models are

i. Economic Order Quantity(EOQ) model
ii. Production Order quantity model
iii. Quantity Discount model

We will now consider deterministic inventory models in which we assume that the rate of demand fort the item is constant or nearly constant. Later we will consider probabilistic inventory models in which the demand for the item fluctuates and can be described only in probabilistic terms

5.6.1: Economic Order Quantity (EOQ) Model

The **Economic Order Quantity(EOQ)** model is applicable when the demand for an item has a constant, or nearly constant, rate and when the entire quantity ordered arrives in the inventory at one point in time. The constant demand rate assumption means that the same number of units is taken form inventory each period of time such as 5 units every day, 25 units every week, 100 units every 4 week period, and so on.

Is the calculated quantity $Q^*(Q_{opt})$ which minimizes the balance of costs between inventory holding cost and re-order cost. For small values of Q*, more frequent ordering will be necessary and hence costs will be high, while large values of Q* will increase the quantity in store and therefore increase the storage cost. The problem is to determine the value of Q* that minimizes the sum of the order cost and storage cost.

Assumptions.

i. Demand is known, constant and independent
ii. Lead time is known and constant
iii. Quantity discount is not possible.
iv. Receipt of inventory is instantaneous and complete. In other words, the inventory from an order arrives in one batch at one time.
v. The only variable costs are the cost of setting up or placing order (setup cost) and the cost of holding or storing inventory over time (holding or carrying cost)
vi. Stock outs (shortages) can be completely avoided if orders are placed at the right time.

With these assumptions, it is apparent that the significant costs are demand cost, setup(ordering) cost, and holding(carrying)cost. Thus, if we minimize the setup and holding costs, we will be minimizing total cost, which is the objective of most inventory models.

If the cost of placing order is C, the total order cost is simply the number of orders made multiplied by C. If D is the demand over a specified time period, then the number of orders made is D/Q , and the order cost is C x D/Q, and demand cost is DC

To calculate the storage/inventory cost it is assumed that the cost of holding one unit in stock foe a specified time period is known. This cost is represented by H. As the amount in stock varies, the average stock level is Q/2. Hence, the storage cost is H x Q/2.

The total cost (TC) =Annual purchase + Annual ordering cost + Annual holding cost

$$= DC + \frac{DS}{Q} + \frac{Q}{2}H$$

Illustration: Imagine that you work for Ghana Computers, and you have been asked to decide on the optimum order quantity. You were told that the demand is fairly constant at 5000 units per annum, and it costs $14.40 to place an order. You are also told that the storage cost of holding one unit of computer per annum is $10.

In order to investigate how inventory cost vary with order size, you decide to work out the order and storage cost for different order quantities as shown below.

Order Quantity	Order cost(C_0) =CD/Q	Storage cost(C_s) = HQ/2	Total cost= $C_0 + C_s$
20	14.4*5000/20=3600	10*20/2=100	3600+100=3700
40	1800	200	2000
60	1200	300	1500
80	900	400	1300
100	720	500	1220
120	600	600	1200
140	512.3	700	1214.3
160	450	800	1250
180	400	900	1300
200	360	1000	1360

From the table, it is apparent that an order quantity of 120 gives the lowest total cost at $12000 per annum.

In general, we have the following:

i. Annual setup cost = Number of orders x setup cost per order

$$= \frac{\text{Annual demand}}{\text{Number of units in each order}} \times \text{setup cost per order}$$

$$= \frac{D}{Q} \times S = \frac{DS}{Q}$$

ii. Annual Holding Cost = Average Inventory Level x Holding cost per unit per annum

$$= \frac{\text{Order quantity}}{2} \times \text{Holding cost per unit per annum}$$

$$= \frac{Q}{2} \times H = \frac{QH}{2}$$

This is the same as the storage cost

Total cost (TC) = Demand + Setup cost + holding cost $= DC + \frac{DS}{Q} + \frac{QH}{2}$

$$= DC + \frac{DS}{Q} + \frac{Q}{2}H$$

Differentiating with respect to TC and equating to zero and solving for Q (which we call the Economic Order Quantity, Q*) we have the following:

$$Q^* = \sqrt{\frac{2DS}{H}}$$

Alternatively we use the fact that at the optimum, Setup cost = holding cost

$$\frac{DS}{Q} = \frac{QH}{2}$$

Solving for Q we get

$$Q^* = \sqrt{\frac{2DS}{H}}$$

Using this formula for the above illustration, we have the following:

$Q^* = \sqrt{(2)(14.4)(5000)/10} = 120$ units

iii. Expected number of orders (N) = Demand/Order quantity = D/Q

iv. Expected time between orders(T) = $\dfrac{\text{Number of working days per annum}}{\text{Expected number of orders}}$

Simple inventory models assume that receipt of an order is instantaneous. In other words, they assume that:

i. a firm will place order when the inventory level for that particular item reaches zero, and,
ii. it will receive the ordered items immediately.

'When to order decision' is expressed in terms of a re-order point (level)- that is the inventory level at which an order should be placed. We make use of the following formulae:

ROP = Demand per day x Lead time for an order in days

$$= \frac{\text{Annual demand}}{\text{No. of working days in a year}} \times \text{Lead time for an order in days}$$

$$= d \times T$$

Example 1: Electronic Assemblier Inc. has a demand for 8000 VCR's per year. The firm operates a 250-day working year. On average, delivery of an order takes 3 working days. Calculate the re-order point.

Solution:

Demand per day (d) $= \dfrac{\text{Demand per year(D)}}{\text{No. of working days in year}} = \dfrac{8000}{250} = 32\text{units} = d$

ROP = Demand per day x Lead time for a order in days

$$= \frac{\text{Annual demand}}{\text{No. of working days in a year}} \quad x \quad \text{Lead time for an order in days}$$

$$= d \ x \ T$$

$\therefore$ ROP = 32 x 3 = 96units.

Thus, when inventory drops to 96 units, an order should be placed. The order will arrive 3 days latter, just as the firm's stock is depleted

Example 2: Find the Economic Order Quantity and the Re-order point given the following.

Annual demand = 1000 units

Ordering cost = $5 per order

Lead time = 5days

Holding cost = $1.25 per unit per year

Cost per unit = $12.50

Number of days in a year = 365 days

Solution:

a. $Q^* = \sqrt{\frac{2DS}{H}} = \sqrt{\frac{2(1000)(5)}{1.25}} = 89.4$ units

b. The re=order point is $ROP = \frac{1000(5)}{365}$

$$= 13,7 \text{ units}$$

c. Total cost(TC) = DC+DS/Q+QH/2

$$= 1000(12.50) + 1000(5)/89 + 89(1.25)/2$$

$$= \$12.61$$

Example 3: Sharp Inc; a company that markets painless hypodermic needles to hospitals would like to reduce its inventory cost by determining the optimal number of hyperdemic needles to obtain per order. The annual demand is 1000 units, the setup cost or ordering cost is $10 per order, and the holding cost per unit per year is $0.50. Calculate:

a. The optimal number of units per order (EOQ).

b. The number of orders (N).

c. The expected time between orders (T).

d. The ROP. (Assume a 250 – day working year).

Solution:

d. $Q^* = \sqrt{\frac{2DS}{H}} = \sqrt{\frac{2(100)(10)}{0.50}} = 200$ units

b. $N = \frac{\text{Demand}}{\text{Order quantity}} = \frac{D}{Q^*} = \frac{1000}{200} = 5$ order per annum

c. $T = \frac{\text{No. of Working days per year}}{\text{Expected no. of order(N)}} = \frac{250}{5} = 50$ days between orders

d. Demand per day (d) $= \frac{\text{Demand per year(D)}}{\text{No. of working days in year}} = \frac{1000}{250} = 4$units = d

ROP = Demand per day x Lead time for a order in days

$= \frac{\text{Annual demand}}{\text{No. of working days in a year}} \times$ Lead time for an order in days

$= d \times T = 4 \times 50 = 200$ units.

Thus, when inventory drops to 200 units, an order should be placed. The order will arrive 50 days latter, just as the firm's stock is depleted

5.6.2: Economic Production Lot Size Model

In the previous inventory model, we assumed that the entire inventory ordered was received at one tine. There are at times, however, when the firm may receive its inventory over a period of time. Such cases require a different model, one that does not require the instantaneous receipt assumption.

This model is an application under two situations:

i. When inventory continuously flows or builds up over a period of time after an order has been placed.

ii. When units are produced and sold simultaneously.

Production Order Quantity model is an economic Order quantity technique applied to production orders. This model makes use of the following relation.

$$TC = DC + \frac{DS}{Q} + (P - d)\frac{QH}{2p}$$

Differentiating with respect to Q and setting the equation equal to zero we obtain

$$Q^*{}_p = \sqrt{\frac{2DS}{H\left(1 - \frac{d}{p}\right)}}$$

Where:
Q^*_p = Optimum quantity

Q = Number of units per day

H = Holding cost per unit per year.

p = Daily production rate.

d = Daily demand rate or usage rate.

t = length of the production run in days.

The following terminologies are relevant in the derivation of the Production Order Quantity.

i. Annual Inventory Cost = Average inventory level x Holding cost per unit per year.

ii. Annual Inventory Level = $\frac{\text{Maximum inventory level}}{2}$

iii. Maximum Inventory Level = Total production during the production run minus total used during the production run = pt – dt

iv. Total produced, Q = pt

$$\therefore t = \frac{Q}{p}$$

$\therefore$ Maximum Inventory level = $p\left(\frac{Q}{p}\right) - d\left(\frac{Q}{P}\right)$

$$= Q - \frac{d}{p}Q$$

$$= Q\left(1 - \frac{d}{p}\right)$$

$$= \frac{MaxInventoryLevel}{2} \times HoldingCostPerUnitPerAnum.$$

v. Annual Inventory Holding Cost

$$= \frac{MaxInventoryLevel}{2} \times HoldingCostPerUnitPerAnum.$$

$$= \frac{Q}{2}\left(1 - \frac{d}{p}\right) \times H$$

$$= \frac{QH}{2}\left(1 - \frac{d}{p}\right)$$

vi. Setup cost $= \frac{DS}{Q}$

Note: An alternative way to obtain Q^*_p is to equate setup cost to Holding cost and solve for Q(call this Q*)

Example 1: Nathan Manufacturing Inc. makes and sells specially hubcaps for the retail automobile market. Nathan's forecast for its hubcaps is 100 units for the following year, with an average daily demand of 4 units. The company produces 8 units per day but uses 4 units per day. The setup cost (ordering cost) is $10 per order and the holding cost per unit per year is $0.50.

Given the above information, calculate the optimum number of units to produce. (Take working days in a year to be 250 days).

Solution:

Annual Demand(D) = 1000 units

Setup cost(S) = $10

Holding cost(H) = $0.50 per unit per year

Daily production rate(p) = 8 units daily

Daily demand rate(d) = 4 units daily

$$Q^*_p = \sqrt{\frac{2DS}{H\left(1-\frac{d}{p}\right)}} = \sqrt{\frac{2\times 1000\times 10}{0.5\times\left(1-\frac{4}{8}\right)}} = 283\,\text{hubcaps}$$

5.6.3: Quantity Discount Model

To increase sales, many companies offer quantity discounts to their customers. A quantity discount is simply a reduced price (p)for an item when it is purchased in large quantities. In this case the cost of the good is added to the order and the storage costs. Thus, total cost is given as:

Total cost(TC) = Setup cost + Holding cost + Product cost = DS/Q + QH/2 + PD

Consider the data in the table below.

Discount number	Discount quantity	Discount (%)	Discount price($)
1	0 –500	0	5.00
2	501-1000	4	5.00*96%=4.80
3	1001 -1500	5	5.00*95%= 4.75
4	1501 - 2000	10	5.00*90%= 4.50

As can be seen, the normal price of the item is $5.00. When 501 – 1000 units are ordered at one time, the price per unit drops to $4.80, when the quantity ordered at one time is between 1001 and 1500 the price drops further to \$4.75 and when the quantity ordered at one time is between 1501 and 2000 the price drops further to \$4.50

The overall objective of any inventory control model is to minimize total cost. The question is: 'what order quantity will minimize total cost given the discount prices'?
The procedure is as follows:

1. For each discount, calculate a value for the optimal order size Q* using the equation

$$Q^* = \sqrt{\left(\frac{2DS}{IP}\right)}$$

**Note that Holding cost, H, in the EOQ formula is replaced by IP.

2. For any discount, if the order quantity is too low to qualify for the discount, adjust the order quantity upward to the lowest quantity that will qualify for the discount.

3. Using the total cost equation, $TC = DS/Q + QH/2 + PD$, compute a total cost for every Q* determined in steps one and two. If you have to adjust Q* upward because it is below the allowable quantity, be sure to use the adjusted value for Q*.

4. Select the Q* that has the lowest cost as computed in step three. This is the quantity that will minimize the total cost.

<u>Example 1</u>

Koh's discount store stocks toy race cars. Recently, the store has been given a quantity discount schedule for these cars. The quantity discount schedule is shown in the table below.

Discount number	Discount quantity	Discount (%)	Discount price
1	0 – 999	0	5.00

2	1000 – 1999	4	4.80
3	2000 and more	5	4.75

Furthermore, the ordering cost is $49.00 per order, annual demand is 5000 race cars, and inventory carrying charge as a percentage of cost, I, is 20%. What order quantity will minimize the total inventory cost?

Solution:

The first step is to compute Q* for every discount This is done as follows.

$D = 5000$

$S = 49.00$

$I = 20\% = 0.2$

$P_1 = 5, \quad P_2 = 4.80, \quad P_3 = 4.75,$

$$Q^*_1 = \sqrt{\frac{2 \times 500 \times 49}{0.2 \times 5}} = 700$$

$$Q^*_2 = \sqrt{\frac{2 \times 500 \times 49}{0.2 \times 4.8}} = 714$$

$$Q^*_3 = \sqrt{\frac{2 \times 500 \times 49}{0.2 \times 4.75}} = 718$$

The second step is to adjust upward those values of Q* that are below the allowable discount range. The adjustments are done as follows:

- Since Q^*_1 is between 0 and 999, it will not be adjusted.
- Because Q^*_2 is below the allowable range of 1000 to 1999, it must be adjusted to 1000.
- Because Q^*_3 is below the allowable range of 2000, it must be adjusted to 2000.

The adjusted order quantities are as follows:

$Q^*_1 = 700$ (Not adjusted)

Q^*_2 = 1000 (Adjusted)

Q^*_3 = 2000 (Adjusted)

He third step is to use the total cost equation and compute a cost for each order quantity.. This is shown in the table below.

Discount number	Unit price	Order quantity	Production cost(PD)	Ordering cost(DS/Q*)	Holding cost(HQ/2)	Total cost (4)+(5)+(6)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	5.00	700	25000	350	350	25700
2	4.80	1000	24000	245	480	24725
3	4.75	2000	23750	122.5	950	24822.5

The fourth step is to select that order quantity with the lowest cost. The order quantity of 1000 toy cars will minimize the total inventory cost.

Example 2

Problem 1. Zinc Ltd. offers a 7% discount on orders of 400 rolls or more, a 10% discount for 900 rolls or more, a 12% discount for 1000 rolls or more, and a 15% discount for 2000 rolls or more her regular customer, Mike who operates a wholesale shop. The unit price is $3.20 and the setup cost is $20. Inventory carrying charge as a percentage of cost, I, is 25%. What order quantity will minimize the total inventory cost?

Solution:

Discount Number	Discount quantity	Discount (%)	Discount Price (&)
1	Less 400	-	3.20
2	400 or more	7	3.20*.93 =2.976
3	900 or more	10	3.20*.90 = 2.88
4	2000 or more	15	3.20*.90 = 2.72

$Q_1^* = \sqrt{(2*1092*20) / (.25*3.20)} = 233.67 = 234$

$Q_2^* = \sqrt{(2*1092*20) / (.25*2.976)} = 242.30 \Rightarrow$ Not feasible. $\therefore$ $Q_2^* = 400$

$Q_3^* = \sqrt{(2*1092*20) / (.25*2.88)} = 246.31 \Rightarrow$Not feasible. $\therefore$ $Q_3^* = 900$

$Q_4^* = \sqrt{(2*1092*20) / (.25*2.72)} = 253.45 \Rightarrow$Not feasible. $\therefore$ $Q_4^* = 2000$

Discount Number	Unit Price ($)	Order Quantity	Production Cost	Ordering Cost	Holding Cost	Total
1	3.20	234	1092*3.20	1092*20/234	.25*3.2*.5	3681
2	2.976	400	1092*2.976	1092*20/400	25*2.976*.5	3453
3	2.88	900	1092*2.88	1092*20/900	25*2.88*.5	3493
4	2.72	2000	1092*2.72	1092*20/2000	25*2.72*.5	3681

Comparing the total cost for orders for 234, 400, 900, and 2000, the lowest total cost is $3453. Therefore, Nick should order 400 rolls at a time.

SELF ASSESSMENT QUESTIONS

Question 1

Suppose that R&D Beverage Company has a soft drink product that has a constant annual demand rate of 3600cases. A case of the soft drink cost R & D Company$3.00. Ordering costs are $2 per order and holding costs are 255% of the value of the inventory. R&D has 250 working days per year, and lead time is 5 days. Identify the following aspects of the inventory policy:

i. Economic Order Quantity
ii. Reorder Point
iii. Cycle Time
iv. Total annual cost

Question 2

A general property of the EOQ inventory model is that total inventory holding cost and total ordering cost are equal at the optimum solution. Use the data in Question 1 above to show that the result is true.

Question 3

Consider the data in the table below.

Discount number	Discount quantity	Discount (%)	Discount price($)
1	0 –500	0	5.00
2	501-1000	4	5.00*96%=4.80
3	1001 -1500	5	5.00*95%= 4.75
4	1501 - 2000	10	5.00*90%= 4.50

Setup cost is $20. Inventory carrying charge as a percentage of cost, I, is 25%. What order quantity will minimize the total inventory cost?

Question 4

Joe De Ltd. offers a quantity discount of 7% discount on orders of 400 rolls or more, a 10% discount for 900 rolls or more, a 12% discount for 1000 rolls or more, and a 15% discount for 2000 rolls or more her regular customer, Mark Tee who operates a wholesale shop. The unit price of the item is $3.20 and the setup cost is $20. Mark Tee's annual demand is 1092 and inventory carrying charge as a percentage of cost, I, is 25%.

Calculate the order quantity that will minimize inventory cost

UNIT 6

WAITING LINES/ QUEUING THEORY

OUTLINE

OBJECTIVES

SESSION 6.1: INTRODUCTION

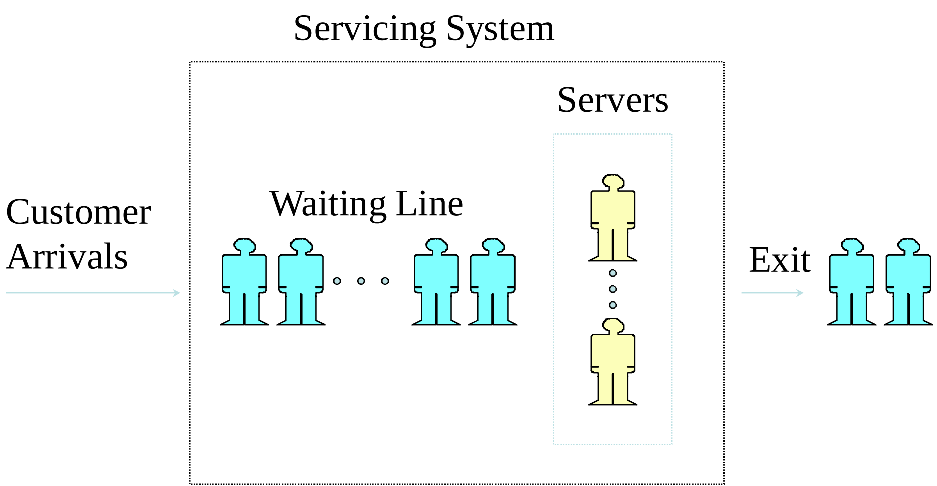
A body of knowledge about waiting lines, often called queuing theory is an important part of operations and a valuable tool for operations managers. Waiting lines may take the form of cars waiting for repair at a shop, customers waiting at a bank to be served, etc.

In many retail stores and banks, management has tried to reduce the frustration of customers by somehow increasing the speed of the checkout and cashier lines. Although most grocery stores seem to have retained the multiple line/multiple checkout system, many banks, credit unions, and fast food providers have gone in recent years to a queuing system where customers wait for the next available cashier. The frustrations of "getting in a slow line" are removed because that one slow transaction does not affect the throughput of the remaining customers.

Walmart and McDonald's are other examples of companies which open up additional lines when there are more than about three people in line. In fact, Walmart has roaming clerks now who can total up your purchases and leave you with a number which the cashier enters to complete the financial aspect of your sale. **Disney** is another company where they face thousands of people a day. One method to ameliorate the problem has been to use queuing theory. It has been proved that throughput improves and customer satisfaction increases when queues are used instead of separate lines. Queues are also used extensively in computing---web servers and print servers are now common. Banks of 800 service phone numbers are a final example I will cite.

Queuing theory leads one directly to the Poisson Distribution, named after the famous French mathematician Simeon Denis Poisson (1781-1840) who first studied it in 1837. He applied it to such morbid results as the probability of death in the Prussian army resulting from the kick of a horse and suicides among women and children. As hinted above, operations research has applied it to model random arrival times.

Components of the Queuing Phenomenon



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SESSION 6.2: POISSON DISTRIBUTION

The Poisson distribution is the continuous limit of the discrete binomial distribution. It depends on the following four assumptions:

1. It is possible to divide the time interval of interest into many small subintervals (like an hour into seconds).
2. The probability of an occurrence remains constant thoughout the large time interval (random).
3. The probability of two or more occurrences in a subinterval is small enough to be ignored.
4. Occurrences are independent.

Clearly, bank arrivals might have problems with assumption number four where payday, lunch hour, and car pooling may affect independence. However, the Poisson Distribution finds applicability in a surprisingly large variety of situations. The equation for the Poisson Distribution is: $P(x) = \mu^{x} \cdot e^{-\mu} \div x!$. The number e in the equation above is the base of the natural logarithms or approximately 2.71828182845904523...

Illustration of the Poisson Distribution

Example 1: On average there are three babies born a day with hairy backs. Find the probability that in one day two babies are born hairy. Find the probability that in one day no babies are born hairy.

Solution: a. $P(2) = 3^2 \cdot e^{-3} \div 2 = .224$ b. $P(0) = 3^0 \cdot e^{-3} = .0498$

Example 2: Suppose a bank knows that on average 60 customers arrive between 10 A.M. and 11 A.M. daily. Thus 1 customer arrives per minute. Find the probability that exactly two customers arrive in a given one-minute time interval between 10 and 11 A.M.

Solution: Let $\mu = 1$ and $x = 2$. $P(2)=e^{-1}/2!=0.3679\div2=0.1839$.

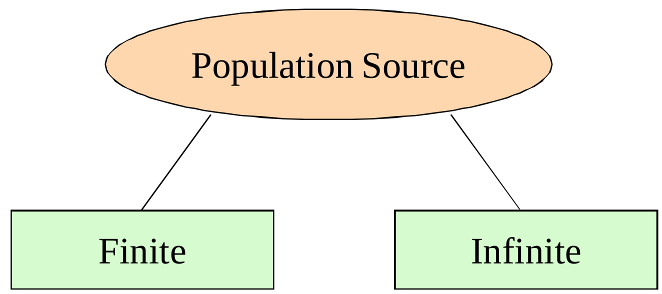
SESSION 6.3: Characteristics of waiting line systems

The three parts of a waiting-line or queuing system are:

1. Arrivals or inputs into the system
2. Queue discipline or the waiting line itself
3. The service facility

1. **Arrivals or inputs into the system**: The input source that generates arrivals or customers for a service has three major characteristics:
   a. **The size of the source population** The population sizes are considered either unlimited(essentially infinite) or limited(finite). When the number of customers or arrivals on one hand at any moment is just a small portion of all potential arrivals, the arrival population is considered **unlimited or infinite.** It could also be considered as a queue in which virtually unlimited number people or items could request the services. Examples of unlimited populations include cars arriving at highway car toll booth, shoppers arriving at a supermarket, and students arriving to register for classes at a large university. A queue in which there are only a limited number of potential users of the service is termed **finite or limited**

Population Sources



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b. **The pattern of arrivals at the queue system**: customers arrive at a service facility either according to some known schedule(for example, 1 patient every 15 minutes or 1 student every half hour) or else they arrive randomly. Arrivals are considered random when they are independent of one another and their occurrence cannot be predicted exactly. Frequently, in queuing problems, the number of arrivals per unit of time can be estimated by probability distribution known as Poisson distribution. For any given arrival time(such as 2 customers per hour or 4 trucks per minute, a discrete Poisson distribution can be established by using the formula:

$$P(x)= \frac{e^{\lambda}\lambda^{x}}{x!} \qquad X = 0, 1, 2, \ldots$$

Where:

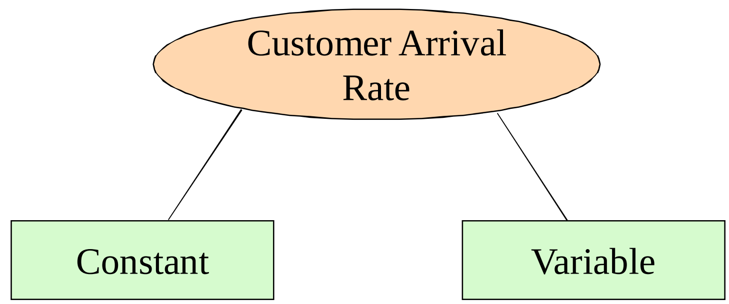
$P(x)$ = probability of x arrivals

X= number of arrivals per unit of time

λ= average arrival rate

e = 2.7183

Customer Arrival



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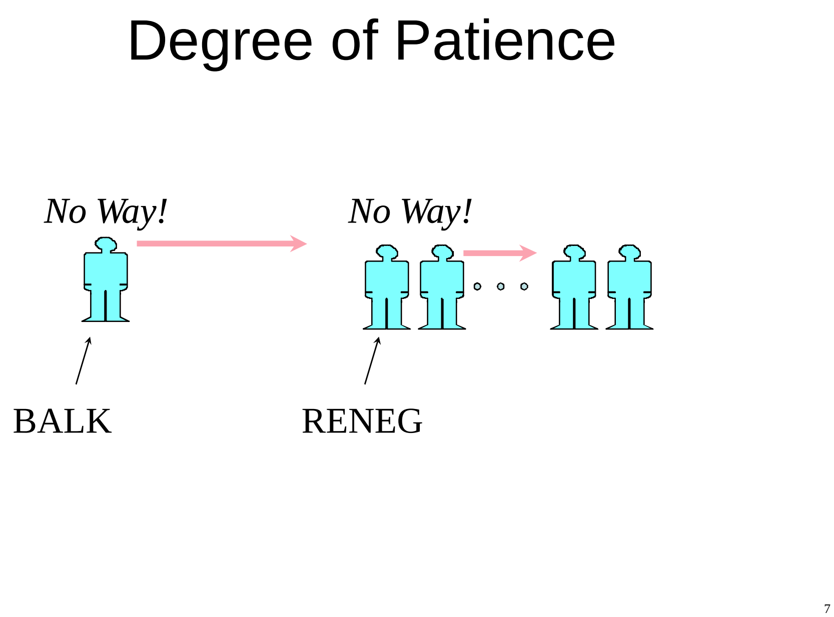
c. **The behaviour of arrivals.** Most queuing models assume that an arriving customer is a patient customer. Patient customers are people or machines that wait in the queue until they are served and do not switch between lines. Unfortunately, life is complicated by the fact that people have been known to **balk or to renege**. Customers who balk refuse to join the waiting line because it is too long to suit their needs or interest. Reneging customers are those who enter the queue but then become impatient and leave without completing their transaction. Actually, both of these situations just serve to highlight the need for queuing theory and waiting-line analysis.



2. **Queue discipline or the waiting line itself:** The waiting line itself is the second component of a queuing system. The length of a line can be ether limited or unlimited. A queue is limited when it cannot, either by law or because of physical restrictions, increase to an infinite length. A barbershop, for example, will have only a limited number of waiting chairs. Queuing models are treated in this module under an assumption of unlimited queue length. A queue is unlimited when its size is unrestricted, as in the case of toll booth serving arriving automobiles. A second waiting-line characteristics deals with queue discipline. This refers to the rule by which customers in the line are to receive service. Most systems use a queue discipline known as **first-in, first-out (FIFO-a queuing discipline where the first customers in line receive first service) rule.** In a hospital emergency room or an express checkout line at a supermarket, however, various assigned priorities may pre-empt FIFO. Patients who are critically injured will move ahead in treatment priority over patients with broken fingers

3. **The service facility:** The third part of any queuing system is the service facility. Two basic properties are important.

a. **The configuration of the service system:** Service systems are usually classified in terms of their number of channels(for example, number of servers) and number of phases(for example, number of service stops that must be made). **A single-channel queuing system** with one server, is typified by the drive-in bank with only one open teller. If on the other hand, the bank has several tellers on duty, with each customer waiting in one common line for the first available teller on duty, then we would have **multiple-channel queuing system**. Most banks today are multichannel service systems, as are most large shops, airline ticket counters, and post offices. In a **single-phase system**, the customer receives service from only one station and then exits the system. A fast food restaurant in which the person who takes your order also brings your food and takes your money is a single-phase system. So is a driver's license agency in which the person taking your application also grades your test and collects your license fee. However, say, the restaurant requires you to place order at one station, pay at a second station, and pick up your food at a third station. In this case, it is **multiphase system.** Likewise, if the driver's license agency is large or busy, you will probably have to wait in one line to complete your application(the first service stop), queue again to have your test graded and finally go to a third counter to pay your fee. To help you relate the concepts of channels and phases, see the chart below.

Line Structures

	Single Phase	Multiphase
Single Channel	One-person barber shop	Car wash
Multichannel	Bank tellers' windows	Hospital admissions

6

b. **The pattern of service times:** Service patterns are like arrival patterns in that they may be either constant or random. If service time is constant, it takes the same amount of time to take care of each customer. This is the case in a machine-performed service operation such as an automatic car was. More often, service times are randomly distributed. In many cases, we can assume that random service times are described by negative exponential probability distribution..

SESSION 6.4: Measuring the queue's performance

Queuing models help managers make decisions that balance service costs with waiting line cost. Queuing analysis can obtain many measures of a waiting-line system's performance including the following:

i. Average time that each customer or objects spends in the queue.
ii. Average queue length
iii. Average time that each customer spends in the system(waiting time plus service time)
iv. Average number of customers in the system
v. Probability that the service facility will be idle
vi. Utilization factor for the system
vii. Probability of a specific number of customers in the system

SESSION 6.5: Queuing Model

All the queuing models that we shall look at have the following characteristics:

i. Poisson distribution arrivals
ii. FIFO discipline
iii. A single-service phase

The different models are summarized in the table below

Model	Name	No. of channels	No. of Phases	Service time pattern	Population size	Queue discipline
A	Single-Channel (M/M/1)	Single	Poisson	Exponential	Unlimited	FIFO
B	Multichannel (M/M/S)	Multi	Poisson	Exponential	Unlimited	FIFO
C	Constant service (M/D/1)	Single	Poisson	Constant	Unlimited	FIFO
D	Limited population	Single	Poisson	Exponential	Limited	FIFO

1. Model A: A single-Channel Queuing model with Poisson arrivals and Exponential service time

The most common case of queuing problem involves the single-channel, or single – server, waiting-line. In this situation, arrivals from single-line to be served by a single station. We assume the following conditions exist in this type of system:

a. Arrivals are served on first-in, first-out(FIFO) basis, and every arrival waits to be served, regardless of the length of the line or queue.
b. Arrivals are independent of preceding arrivals, but the average number of arrivals(arrival rate) does not change.
c. Arrivals re described by a Poisson probability distribution and come from an infinite(or very, very large) population.
d. Service times vary from one customer to the next and are independent of one another, but their average rate is known.
e. Service times occur according to the negative exponential probability distribution.
f. The service rate is faster than the arrival rate.

When these conditions are met, the series of equations shown below can be developed. Example 1 and 2 illustrate how model A(which in technical journals is known as M/M/1 model) may be used.

Queuing formulae for model A: A simple system , also called M/M/1

λ = Mean number of arrivals per time period

μ = Mean number of people or items served per time period

(1). L_s = Average number of units(customers) in the system $= \dfrac{\lambda}{\mu - \lambda}$

(2). W_s = Average time a unit(customer) spends in the system

(waiting time plus service time) $= \dfrac{1}{\mu - \lambda}$

(3). L_q = Average number of units in the queue $= \dfrac{\lambda^2}{\mu(\mu - \lambda)} =$

(4). W_q = Average time a unit spends waiting in the queue $= \dfrac{\lambda}{\mu(\mu - \lambda)}$

(5). P_0 = Probability of 0 units in the system (that is the service unit is idle) $= 1 - \dfrac{\lambda}{\mu}$

(6). ρ = Utilization factor for the system $= \dfrac{\lambda}{\mu}$

$P_{n>k}$ =Probability of more than k units in the system, where K is the number of

units in the system $= \left(\dfrac{\lambda}{\mu}\right)^{k+1}$

Example 1

Jones, the mechanic at Golden Muffler Shop, is able to install new mufflers at an average rate of 3 per hour(or about 1 every 20 minutes), according to a negative exponential distribution. Customers seeking this service arrive at the shop on the average of 2 per hour, following a Poisson distribution. They are served on a first-in, first-out basis and come from a very large population of possible buyers.

From the description above, we deduce the operating characteristics of Golden Muffler's queuing system

λ = Mean number of arrivals per time period = 2 cars arriving per hour

μ = Mean number of people or items served per time period = 3 cars serviced per hour

L_s = Average number of units(cars) in the system $= \frac{\lambda}{\mu - \lambda} = \frac{2}{3-2} = 2$

W_s = Average time a unit(car) spends in the system (waiting time plus service time)

$= \frac{1}{\mu - \lambda}$

$= \frac{1}{3-2} = 1$

L_q = Average number of cars in the queue $= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{2^2}{3(3-2)} = \frac{4}{3}$

W_q = Average time a car spends waiting in the queue $= \frac{\lambda}{\mu(\mu-\lambda)} = \frac{2}{3(3-2)} = \frac{2}{3}$

P_0 = Probability of 0 units in the system (that is the service unit is idle) $= 1 - \frac{\lambda}{\mu}$

ρ = Utilization factor for the system $= \frac{\lambda}{\mu} = \frac{2}{3}$ =66.7% of time the mechanic is busy

$P_{n>k}$ = Probability of more than k units in the system, where K is the number of units in the system $= \left(\frac{\lambda}{\mu}\right)^{k+1}$

k	$P_{n>k} = (2/3)^{k+1}$	Comments
0	0.667	Note that this is equal to 1-P_0 = 1-0.33 = 0.667
1	0.444	
2	0.296	
3	0.198	This implies that there is 19.9% chance that more than 3 cars are in the system
4	0.132	
5	0.088	
6	0.058	
7	0.039	

Model B: Multiple-Channel Model

This is a system in which two or more servers or channels are available to handle arriving customers. We still assume that customers awaiting service from one single line and then proceed to the first available server. Multi-channel, single-phase waiting line are found in many banks today: a common line is formed and the customer at the head of the line proceeds to the first free teller(Appendix).

The multi-channel system presented in Appendix again assumes that arrivals follow a Poisson Probability distribution and that service times are exponentially distributed. Service is first-come, first-served, and all servers are assumed to perform at the same rate. Other assumptions listed for the single-channel model also apply.

Queuing formulae for Model B: A simple system, also called M/M/S

The equations are more complex than those in the single-channel model.

SESSION 6.6: Suggestions for Managing Queues

1. **Determine the acceptable waiting time for your customers**. How long do your customers expect to wait? Set operational objectives based on what is acceptable.
2. **Try to divert your customers' attention when waiting**. Install distractions that entertain and physically involve the customer. Providing music, video or some other form of entertainment may help to distract the customers from the fact they are waiting.
3. **Inform your customers of what to expect**. This is especially important when the waiting time will be longer than normal. Tell them why the waiting time is longer than normal and what you will do to alleviate the queue.
4. **Keep employees not serving the customers out of sight.** Nothing is more frustrating to someone waiting in line than to see employees, who potentially could be serving those in line, working on other activities.
5. **Segment customers by personality types or nature of service.** If a group of customers needs something that can done very quickly, give them a special line so that they do not wait for the slower customers.

6. **Train your servers to be friendly.** Never underestimate the power of a friendly server. Greeting the customer by name, or providing other special attention, can go a long way toward overcoming the negative feeling of a long wait.
7. **Modify customer arrival behavior.** Encourage customers to come during the slack periods. Inform customers of times when they usually would not have to wait. Also tell them when the peak periods are.
8. **Adopt a long-term perspective getting rid of the queues.** Develop plans for alternative ways to serve your customers. Where appropriate, develop plans for automating or speeding up the process in some manner.

SELF ASSESSMENT QUESTIONS

QUESTION 1

There is only 1 copying machine in the student hostel of the polytechnic. Students arrive at the rate of 40 per hour(according to a Poisson distribution). Copying takes an average rate of 40 seconds or 90 per hour(according to an exponential distribution). Compute the following:

a. The percentage of time that the machine is used.
b. The average length of the queue.
c. The average number of students in the system
d. The average time spent waiting in the queue
e. The average time in the system

QUESTION 2

Sam, a vertenary doctor, is running a rabies-vaccination clinic for dogs in the local grade school. Sam can “shoot” a dog every 3 minutes. It is estimated that the dogs will arrive independently and randomly throughout the day at a rate of 1 dog every 6 minutes according to a Poisson distribution. Also assume that Sam’s shooting times are exponentially distributed. Compute the following:

i. The probability the Sam is idle.
ii. The proportion of time that Sam is busy
iii. The average number of dogs being vaccinated and waiting to be vaccinated
iv. The average number of dogs waiting to be vaccinated
v. The average time a dog waits before getting vaccinated
vi. The average amount of time a dog spends waiting in line and being vaccinated

QUESTION 3

Calls arrive at John DeBruzzi's hotel switchboard at a rate of 2 per minute. The average time to handle each is 20 seconds. There is only 1 switchboard operator at the current time. The Poisson and exponential distributions appear to be relevant in this situation

a. What is the probability that the operator is busy?
b. What is the average time that a caller must wait before reaching the operator?
c. What is the average number of calls waiting to be answered?

QUESTION 4

The Charles Leitle Electronics Company retains a service crew to repair machine breakdowns that occur on average of 3 per day(approximately Poisson in nature). The crew can service an average of 8 machines per day, with time distribution that resembles the exponential distribution

i. What is the utilization rate of this service?
ii. What is the average downtime for a broken machine?
iii. How many machines are waiting to be serviced at any given time?
iv. What is the probability that:
    a. more than one machine is in the system
    b. more than two are broken and waiting to be repaired or being serviced?
    c. More than three are broken and waiting to be repaired or being serviced?
    d. More that four are broken and waiting to be repaired or being serviced?

QUESTION 5

Gab Moss manages a large Mongomery, Alabama, movie theatre complex called Cinema I, II, III, and IV. Each of the four auditorium plays a different film: the schedule staggers starting times to avoid the large crowds that would occur if all 4 movies started at the same time. The theatre has a single ticket booth and a cashier who can maintain an average service rate of 280 patrons per hour. Service times are assumed to follow an exponential distribution. Arrivals on a normally active day are Poisson-distributed and average 210 per hour.

In order to determine the efficiency of the current ticket operation, Gab wishes to examine several queue –operating characteristics.

i. Find the average number of moviegoers waiting to purchase a ticket
ii. What percentage of the time is the cashier busy?
iii. What is the average time that a customer spends in the system?
iv. What is the average time spent waiting in line to get the ticket window?
v. What is the probability that there are more than two people in the system, more than 3, more than 4?

QUESTION 6

The cafeteria line in the university student center is a self-serve facility in which in which students select the items they want and then form a single line to pay the cashier. Students arrive at a rate of about 4 per minute according to a Poisson distributing. The single cashier takes about 12 seconds per customer, following an exponential distribution

a. What is the probability that there are more than 2 students in the system? More than 3 students, more than 4?
b. What is the probability that the system is empty?
c. How long will the average student have to wait before reaching the cashier?
d. What is the average number in the system?
e. If a second cashier is added and works at the same pace, how will the operating characteristics computed in parts (b), (c), (d), and (e)

change?(Assume customers wait in a single line and go to the first available cashier).

QUESTION 7

The administrator at a large hospital emergency room faces the problem of providing treatment for patients who arrive at different rates during the day. There are 4 doctors available to treat patients when needed. If not needed, they can be assigned other responsibilities(such as doing lab tests, reports, X-rays diagnoses) or else rescheduled to work at other hours.

It is important to provide quick and responsive treatment, and the administrator feels that, on the average, patients should not have to sit in the waiting area more than 5 minutes before being see by a doctor. Patients are treated on a first-come, first served basis, and see the first available doctor after waiting in the queue. The arrival pattern for a typical day is as follows:

Time	Arrival rate
9am – 3pm	6 patients/hour
3pm – 8pm	4 patients/hour
8pm - midnight	12 patients/hour

Arrivals follow a Poisson distribution, and treatment times, 12 minutes on the average, follow the exponential pattern.

How many doctors should be on duty during each period to maintain the level of patient care expected?

QUESTION 8

Two technicians, working as a team, monitor a group of 5 computers that run an automated manufacturing facility. It takes an average of 15 minutes (exponentially distributed) to adjust a computer that develops a problem. Computers run for an average of 85 minutes(Poisson-distributed) without requiring adjustments. Determine the following:

a. The average number of computers waiting for adjustments
b. The average number being adjusted
c. The average number of computers not in working order.

QUESTION 9

1. What is a waiting line problem? What are the components in a waiting – line system?
2. Briefly describe three situations in which the FIFO discipline rule is not applicable in queuing analysis
3. Provide 4 examples of four situations in which there is limited , or finite waiting line
4. Most banks have changed from having a line in front of each teller to a situation whereby one line feeds all tellers. Which system is better? Why?

SIMULATION

1. Introduction

Simulation analysis is a natural and logical extension to the analytical and mathematical models inherent in operations research. There are many situations which cannot be represented mathematically due to the stochastic nature of the problem, the complexity of problem formulation, or the interactions needed to adequately describe the problem under study. For many situations defying the mathematical formulation, simulation is the only tool that might be used to obtain relevant answers.

Typical business examples are: most queuing systems other than the very simplest, inventory control problems, production planning problems, etc.

Definition 1: *A model is any representation (physical or abstract)of a real thing, event or circumstances*

Definition 2: *Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and nature of a complex real world system over extended period of time.*

SIMULATION METHODOLOGY

Problem efinition

Problem definition for purposes of simulation differs little from problem definition for any other tool of analysis. Essentially, it entails specifying the objectives and identifying the objectives and identifying the relevant controllable and uncontrollable variables of the system to be studied. Consider the example of a fish market. The objective of the market's owner is the maximization of profit on sales of fish. The relevant controllable variable (that is, under the control of the decision maker) is the ordering rule; the relevant uncontrollable variables are the daily demand levels for fish and the amount of fish sold. Other possible objectives could also be specified, such as to maximize profit from the sale of lobsters or to maximize sales revenue.

Constructing a Simulation Model

A feature that distinguishes simulation from other techniques such as linear programming or queuing theory is the fact that a simulation model must be custom built for each problem situation. A linear programming model, in contrast, can be used in a variety of situations with only a restatement of the values for the objective function and constraint equations. There are simulation languages that make the model building easier, however. The unique nature of each simulation model means that the procedures for building and executing a model represent a synthesis of various approaches to simulation and are guidelines rather than rigid rules.

Specification of Variables and Parameters

The first step in constructing a simulation model is determining which properties of the real system should be fixed (called **parameters)** and which should be allowed to vary throughout the simulation run (called **variables**). In the fish market, the variables are the amount of fish ordered, the amount demanded, and the amount sold; the parameters are the cost of the fish and the selling price of the fish. In most simulations, the focus is on the status of the variables at different points in time, such as the number of pounds of fish demanded and sold each day.

Specification of Decision Rules

Decision rules (or operating rules) are sets of conditions under which the behaviour of the simulation model is observed. These rules are either directly or indirectly the focus of most simulation studies. In many simulations, decision rules are priority rules (for example, which customer to serve first, which job to process first). In certain situations these can be quite involved, taking into account a large number of variables in the system. For example, an inventory ordering rule could be stated in such a way that the amount to order would depend on the amount in inventory, the amount previously ordered but not received, the amount backordered, and the desired safety stock.

Specification of Probability Distributions

Two categories of distributions can be used for simulation: empirical frequency distributions and standard mathematical distributions. An empirical distribution is derived from observing the relative frequencies of some event such as arrivals in a line or demand for a product. In other words, it is a custom-built demand distribution that is relevant only to a particular situation. Such distributions have to be determined by direct observation or detailed analysis of records.

Specification of Time-Incrementing Procedure

In a simulation model, time can be advanced by one of two methods: (1) fixed-time increments or (2) variable-time increments. Under both methods of **time incrementing**, the concept of a simulated clock is important. In the fixed-time increment method, uniform clock time increments (such as minutes, hours, or days) are specified, and the simulation proceeds by fixed intervals from one time period to the next. At each point in clock time, the system is scanned to determine if any events are to occur. If they are, the events are simulated, and time is advanced; if they are not, time is still advanced by one unit.

In the variable-time increment method, clock time is advanced by the amount required to initiate the next event. Which method is most appropriate? Experience suggests that the fixed –time increment is desirable when events of interest occur with regularity or when the number of events is large, with several commonly occurring in the same time period. The variable-time increment method is generally desirable, taking less computer run time when there are relatively few events occurring within a considerable amount of time. It ignores time intervals where nothing happens and immediately advances to the next point when some event does take place.

Specifying Values of Variables and Parameters

A variable, by definition, changes in values as the simulation progresses, but it must be given an initial starting values. The value of parameter stays constant, how-ever, it may be changed as different alternatives are studied in other simulations.

Determining Starting Conditions

Determining starting conditions for variables is a major tactical decision in simulation. This is because the model is biased by the set of initial starting values until the model has settled down to a steady state. To cope with this problem, analysts have followed various approaches such as (1) discarding data generated during the early parts of the run, (2) selecting starting conditions that reduce the duration of the warm-up period, or (3) selecting starting conditions that eliminate bias. To employ any of these alternatives, however, the analyst must have some idea of the range of output data expected. Therefore, in one sense, the analyst biases results. On the other hand, one of the unique features of simulation is that it allows judgment to enter into the design and analysis of the simulation; so if the analyst has some information that bears on the problem, it should be included.

Determining Run Length

The length of the simulation run (**run length** or **run time**) depends on the purpose of the simulation. Perhaps the most common approach is to continue the simulation until it has achieved equilibrium. In the fish market example, this would mean that simulated fish sales correspond to their historical relative frequencies. Another approach is to run the simulation for a set period, such as a month, a year, or a decade, and see if the conditions at the end of the period appear reasonable. A third approach is to set run length so that a sufficiently large sample is gathered for purposes of statistical hypothesis testing.

Evaluating Results

The types of conclusions that can be drawn from a simulation depend, of course, on the degree to which the model reflects the real system, but they also depend on the design of the simulation in a statistical sense. Indeed, many analysts view simulation as a form of hypothesizing, with each simulation run providing one or more pieces of sample data that are amenable to formal analysis through inferential statistical methods. Statistical procedures commonly used in evaluating simulation results include analysis of variance, regression analysis, and t-tests.

In most situations, the analyst has other information available with which to compare the simulation results: past operating data from the real system, operating data from the performance of similar systems, and the analyst’s own intuitive understanding of the real system’s operation. Admittedly, however, information obtained from these sources is probably not sufficient to validate the conclusions derived from the simulation. Thus, the only true test of simulation is how well the real system performs after the results of the study have been implemented.

Validation

In this context, *validation* refers to testing the computer program to ensure that the simulation is correct. Specifically, it is a check to sees whether the computer code is a valid translation of the flowchart model and whether the simulation adequately represents the real system. Errors may arise in the program from mistakes in the coding or from mistakes in logic. Mistakes in coding are usually easily found because the program is most likely not executed by the computer. Mistakes in logic, however, present more challenge. In these cases, the program runs but fails to yield correct results.

To deal with this problem, the analyst has three alternatives: (1) have the program print out all calculations and verify these calculations by separate computation, (2) simulate present conditions and compare the results with the existing system, or (3) pick some point in the simulation run and compare its output to the answer obtained from solving a relevant mathematical model of the situation at that point. Even though the first two approaches have obvious drawbacks, they are more likely to be employed than the third, because if we had a relevant mathematical model in mind, we would probably be able to solve the problem without the aid of simulation.

Mounte Carlo Simulation

Any realistic business problem contains probabilistic or random features. For example, arrivals at a store may average 60 per day, but the actual arrival pattern is likely variable, in a corporate planning exercise forecasts of the daily sales will obviously vary according different circumstances, etc.

Because models must behave like the real systems under observation, the models must contain these probabilistic elements. When a system contains elements that exhibit chance in their behaviour, the Monte Carlo method of simulation is used.

Definition 3: Monte Carlo method of simulation is a simulation technique that uses random elements when chance exists in their behaviour.

Steps involved in Monte Carlo Simulation

1. *Setting up a probability distribution for important variables.*
2. *Building a cumulative probability for each variable distribution.*
3. *Establishing an interval of random numbers for each variable.*
4. *Generating random numbers.*
5. *Simulating series of trials.*

The example below will be used to demonstrate the steps

The daily demand for tire at AA Auto Tire over the past 200 days is shown in the table below.

Demand for tire	0	1	2	3	4	5
Frequency	10	20	40	60	40	30

Assuming that past arrival rates will hold in the future, we can convert this demand to a probability distribution by dividing each demand frequency by the total demand. The results are shown below

1	**2**	**3**	**4**	**5**
Demand for tire	Frequency	Probability	Cumulative probability	Random number interval
0	10	0.05	0.05	01 through 05
1	20	0.10	0.15	06 through 15
2	40	0.20	0.35	16 through 35
3	60	0.30	0.65	36 through 65
4	40	0.20	0.85	66 through 85
5	30	0.15	1.00	86 through 00
Total	200	1.00		

Once we have established a cumulative probability distribution for each variable, we must assign a set of number to represent each possible outcome. These are referred to as random number intervals (column 5).

If for example, there is a 5% chance that demand for tires will be 0 units per day, then we will want 5% of that random numbers available to correspond to a demand of 0 units. If a total of 100 two-digit numbers are used in the simulation we could assign a demand of 0 units to the first 5 random numbers:01 02 03 04 and 05. Then a simulated demand of 0 units would be created every time one of the numbers 01 to 05 was drawn.

Random numbers are then generated to. For example if the following random numbers are generated; 42, 85, …, 42 is in the interval 36 to 65 which represents daily demand of 3 tires. Again, 85 is in the interval 66 to 85 which also represents daily demand of 4 tires.

Definition 4: A random number is a series of digits, say two, from 01, 02,… 99, 00, that have been selected by a random process- a process where each number has an equal chance of being selected.

SOLVED PROBLEMS

Problem 1

The daily demand for tire at AA Auto Tire over the past 200 days is shown in the table below.

Demand for tire	0	1	2	3	4	5
Frequency	10	20	40	60	40	30

Simulate the daily demand for tire for a period of 10 days. Use the random numbers 52, 37, 82, 69, 98, 96, 33, 50, 88, 90.

Solution

Step 1. First construct a table of frequency, probability(relative frequency), cumulative frequency, and random number intervals.

1	**2**	**3**	**4**	**5**
Demand for tire	Frequency	Probability	Cumulative probability	Random number interval
0	10	0.05	0.05	01 through 05
1	20	0.10	0.15	06 through 15
2	40	0.20	0.35	16 through 35
3	60	0.30	0.65	36 through 65
4	40	0.20	0.85	66 through 85
5	30	0.15	1.00	86 through 00

Step 2. Use the random numbers generated to simulate.

Day	Random #	Simulated daily demand
1	52	3
2	37	3
3	82	4
4	69	4
5	98	5
6	96	5
7	33	2
8	50	3
9	88	5
10	90	5
		Total = 39

Daily demand = 39/10 = 3.9

Problem 2

Simulate the selection of 10 voters for an opinion poll if overall 25% vote for party X, 30% for part T, 40% for party Z with the remaining abstaining. Use the random numbers:

31246 50584 62321 20018

Solution

Party	Probability	Cumulative probability	Random number interval
X	0.25	0.25	01 – 25
Y	0.30	0.55	26 – 55
Z	0.40	0.95	56 – 95
Abstains	0.05	1.00	96 – 00

Sequence of selection	Random number	Party
1	31	Y
2	24	X
3	65	Z
4	05	X
5	84	Z
6	62	Z
7	32	Y
8	12	X
9	00	Abstains
10	18	X

Problem 3

A wholesaler stocks an item for which demand is uncertain. He wishes to assess two re-ordering policies. That is order 10 units at a reorder level of 10 units, or order 15 at a reorder level of 15 units, to see which is most economical over a 10 day period.

The following information is available.

Demand per day	Probability
4	0.10
5	0.15
6	0.25
7	0.30
8	0.20

Carrying costs $15.00 per unit per day. Ordering costs $50.00 per order. Loss of good-will for each unit out of stock is 430.00. Lead time is 3 days. Opening stack is 17 units.

The probability distribution is to be based on the following random numbers

41 92 05 44 66 07 00 00 14 62 20 07 95 05 79 95 64 26 06 48

Note: The reorder level is physical stock plus any replenishment orders outstanding.

Solution

Demand	Probability	Cum. Probability	Random number interval
4	0.10	0.10	01 – 10
5	0.15	0.25	11 –25
6	0.25	0.50	26 –50
7	0.30	0.80	51 – 80
8	0.20	1.00	81 –00

Day	**Opening stock**	**Demand**	**Closing stock**	**Order cost**	**Carrying cost**	**stock out cost**	**Total cost**
1	17	6	11	$50.00	$165.00		$215.00
2	11	8	3		$45.00		$45.00
3	3	4				$30.00	$30.00
4	15	6	9	$50.00	$135.00		$185.00
5	9	7	2		$30.00		$30.00
6	2	4				$60.00	$60.00
7	15	4	11	$50.00	$165.00		$215.00
8	11	4	7		$105.00		$105.00
9	7	5	2		$30.00		$30.00
10	15+2	7	10	$50.00	$150.00		$200.00
				$200.00	$825.00	$90.00	$1,115.00

Day	Opening stock	Demand	Closing stock	Order cost	Carrying cost	stock out cost	Total cost
1	17	5	12		$180.00		$180.00
2	12	4	8	$50.00	$120.00		$170.00
3	8	8					
4		5				$150.00	$150.00
5	10	7	3	$50.00	$45.00		$95.00
6	3	8				$150.00	$150.00
7		7				$210.00	$210.00
8	10	6	4	$50.00	$60.00		$110.00
9	4	4					
10		6				$180.00	$180.00
				$150.00	$405.00	$690.00	$1,245.00

Problem 4

The number of ships docking at any given night at Tema habour ranges from 0 to 5. The probability of 0, 1, 2, 3, 4, and 5 arrivals are shown in the table below:

Number of arrivals	Probability	Cumulative probability	Random number interval
0	0.13	0.13	01 through 13
1	0.17	0.30	14 through 30
2	0.15	0.45	31 through 45
3	0.25	0.70	46 through 70
4	0.20	0.90	71 through 90
5	0.10	1.00	91 through 00

A study by the dock superintendent reveals that the number of ships unloading also tends to vary day to day. The table below provides information on the daily unloading rate.\

Daily unloading rate	Probability	Cumulative probability	Random number interval
1	0.05	0.05	01 through 05
2	0.15	0.20	06 through 20
3	0.50	0.70	21 through 70
4	0.20	0.90	71 through 90
5	0.10	1.00	91 through 00

The principle of loading at the habour is on a first –in, first-out basis (FIFO). Any ship not unloaded on the day of arrival must wait until the following day. He decides that before requesting for additional crews, he should conduct a simulation study of arrivals, unloading and delays. Conduct a 15-day simulation to find out the following:

a. Average number of ships delayed till the next day.

b. Average nightly arrivals.

c. Average number of ships unloaded each day.

Solution

1	**2**	**3**	**4**	**5**	**6**	**7**
Day	Number delayed from previous day	Random number	Number of nightly arrivals	Total to be unloaded	Random number	Number unloaded
1	-	52	3	3	37	3
2	0	06	0	0	63	**0***[1]
3	0	50	3	3	28	3
4	0	88	4	4	02	1
5	3	53	3	6	74	4
6	2	30	1	3	35	3
7	0	10	0	0	24	**0***[2]
8	0	47	3	3	03	1
9	2	99	5	7	29	3
10	4	37	2	6	60	3
11	3	66	3	6	74	4
12	2	97	5	7	85	4
13	3	65	2	5	90	4
14	1	32	2	3	73	**3***[3]
15	0	00	5	5	59	3
	20		**41**			**39**

a. Average number of ships delayed till the next day = 20 delays/15 days

= 1.33 ships delayed per day

b. Average nightly arrivals = 41 arrivals/15 days

= 2.73 arrivals per night

c. Average number of ships unloaded each day = 39 unloadings/15 days

= 2.60 unloadings per day

<u>Comments:</u>

Column 1: Contains days, i.e from day 1 to day 15.

Column 2: Contains number of ships delayed from the previous day. This is obtained by subtracting column 4 from column 4(column5 – column 4).

Column 3: Contains random numbers. These are used to select the number of nightly arrivals.

Column 4: Contains the number of ships that arrive in the night.. It is determined from column 3.

Column 5: Contains the total number of ships that are to be unloaded. Determined as follows: Column 5 = column 4 + (column 4 – column 7)

Column 6: Contains random numbers. Used to select the number of ships to be unloaded

Column 7: Contains number of ships to be unloaded. Determined from column 6

NOTE:

NOTE 1(0^{*1}) = Three (3) ships are to be unloaded on day 2. However there were no arrivals and no backlog, so no unloading took place.

NOTE 1(0^{*2})= The same situation as noted in note 1

NOTE 3 (3^{*3}) = This time, four (4) ships were to be unloaded but because only 3 were in queue, the number unloaded was three (3).

Problem 5

Higgins Plumbing maintains a of 30-gallons of hot water heater that it sells to homeowners and install for them. The owner Jerry Higgins likes the idea of having a large supply on hand to meet any customer demand. However, he also recognizes that it is expensive to do so. He examines hot water sales over the past 50 weeks and noted the following:

Sales per week	Number of weeks this was sold
4	6
5	5
6	9
7	12
8	8
9	7
10	3
Total	**50**

a. If Higgins maintains a customer supply of 8 hot water heaters in any given week, how many times will he stock out during a 20-week simulation?

b. What is the average number of sales per wee over the 20-week period?

c. Using analytic nonsimulation technique, what is the expected number of sales per week.?

d. How does this compare to the answer in (b)?

Solution:

Heater sales	Probability	Random number interval
4	0.12	01 – 12
5	0.10	13 – 22
6	0.18	23 – 40
7	0.24	41 – 64
8	0.16	65 – 80
9	0.14	81 – 94
10	0.06	95 – 00

Week	Random #	Simulated sales	Week	Random #	Simulated sales
1	10	4	11	08	4
2	24	6	12	48	7
3	03	4	13	66	8
4	32	6	14	97	10
5	23	6	15	03	4
6	59	7	16	96	10
7	95	10	17	46	7
8	34	6	18	74	8
9	34	6	19	77	8
10	51	7	20	44	7

a. with a supply of 8 heaters Higgins will stock out 3 times during the 20 week period(i.e. in weeks 7, 14, and 16).

b. Average sales =Total sales/20 weeks

= 135/20

6.75 per week

c. Using expected values

E(sales) = 0.12(4) +0.10(5) + 0.18(6) + 0.24(7) + 0.16(8) + 0.14(9) + 0.06(10)

= 6.86 heaters

With a longer time simulation, these two approaches will lead to even closer results.

Problem 6

A rural clinic receives a delivery of fresh plasma once each week from a central blood bank. The supply varies according to demand from other clinics and hospitals in the region but ranges between four and nine pints of the most widely used blood type, type O. The number of patients per week requiring this blood varies from 0 to 4, and each patient may need from 1 to 4 pints. Given the following delivery quantities, patient distribution, and demand per patient, what will be the number of pints of blood that will be in excess or short for a six-week period. Use simulation to derive your answer. Consider that plasma is storable and there is currently none on hand.

Delivery quantities		Patient information		Demand per patient	
Pints per week	Probability	Patients per week requiring blood	Probability	Demand	probability
4	0.15	0	0.25	1	0.40
5	0.20	1	0.25	2	0.30
6	0.25	2	0.30	3	0.20
7	0.15	3	0.15	4	0.10
8	0.15	4	0.05		
9	0.10				

Week	Beginning inventory	Quantity demanded		Total blood on hand	Patients needing blood		Quantity needed			Number of pints remaining
		RN	Pints		RN	Patients	Patient	RN	Pints	
1	0	74	7	7	08	3	First	21	1	6
							Second	06	1	5
							Third	71	3	2
2	2	31	5	7	28	1		96	4	3
3	3	02	4	7	72	2	First	12	1	6
							Second	67	2	4
4	4	53	6	10	44	1		23	1	9
5	9	16	5	14	16	0				14
6	14	40	6	20	83	3	First	65	2	18
							Second	34	1	17
							Third	82	3	14

At the end of the sixth week, there were 14 pints of blood on hand.

Exercises

Exercises 1

The number of machine breakdown per day at AA factory is either 0, 1, or 2 with probabilities 0.5, 0.3, or 0.2 respectively. The following random numbers have been generated: 13, 14, 02, 18, 31, 19, 32, 85, 31, and 94. Use the numbers to generate the number of breakdowns for 10 consecutive days. What proportion of these days had at least 1 breakdown?

Answer: 0, 0, 0, 0, 0, 0, 0, 2, 0, 2

Proportion = 2/10 = 1/5

Exercises 2

Marissa grocery store has noted the following figures with regard to the number of people who arrive at the store's three check-out stand and the time it takes to check them out..

Arrivals(mins)	Relative frequency	Service time	Relative frequency
0	0.3	1	0.1
1	0.5	2	0.3
2	0.2	3	0.4
		4	0.2

Simulate the utilization of the check-out stands over 5 minutes using the following random numbers: 07 60 77 49 76 95 51 16

Exercises 3

Kate More sells papers at a newspaper stand for $0.35. The papers cost her $0.25 giving her a $0.10 profit on each paper she sells. From past experience Kate knows that:

- 20% of the time she sells 100 papers
- 20% of the time she sells 150 papers
- 30% of the time she sells 200 papers
- 30% of the time she sells 250 papers

Assuming that Kate believes the cost of a lost sale to be $0.05, and any unsold paper costs her $0.25. Simulate her profit outlook over 5 days if she orders 200 papers for each of the 5 days. Use the random numbers 52, 06, 50, 88, 53

Answer: $20.00, -$15.00, $20.00, $17.50, $20.00

Average = $12.50

Exercises 4

The number of cars arriving at XX cars wash during the last 200 hours of operation is observed to be the following.

Number of cars arriving	Frequency
3 or less	0
4	20
5	30
6	50
7	60
8	40
9 or more	0
Total	200

a. Set up a probability and cumulative probability distribution for the variable of car arrivals.
b. Establish random number intervals for the variable.
c. Simulate 15 hours of car arrivals and compute the average number of arrivals per hour. Use the following random numbers: 52 37 82 69 98 96 33 50 88 90 50 27 45 81 66

Answer.

Number of cars arriving	Frequency	Probability	C. Prob	Random # interval
3 or less	0		0	-
4	20		0.10	01 – 10
5	30		0.25	11 – 25
6	50		0.50	26 – 50
7	60		0.80	51 – 80
8	40		1.00	81 – 00
9 or more	0		0	-
Total	200			

Average number of arrivals per hour = 105/15 = 7 cars

Exercises5

The time between arrivals of the drive-through window of XX fast food restaurant follows the distribution given below. The service time distribution is also given.

Time between arrivals	Probability	Service time	Probability
1	0.2	1	0.3
2	0.3	2	0.5
3	0.3	3	0.2
4	0.2		

a. Use the random numbers provided to simulate the activity of the first five arrivals. Assume that the window opens at 11.00 am and that the first arrival occurs afterward based on the first interarrival generated.

Random numbers for arrivals: 14 74 27 03

Random numbers for service time: 88 32 36 24

b. At what time does the 4th customer leave the system?

Answer

Arrivals	Arrival time	Service time	Departure time
1	11.01	3	11.04
2	11.04	2	11.06
3	11.06	2	11.08
4	11.07	1	11.09

Exercises 6

Tira Thompson owns and operates one of the largest Mercedes-Benz auto dealerships in Washington DC. In the past 36 months, his sales have ranged from a low of 6 new cars to a high of 12 new cars as reflected in the table below. Mr. Thompson believes that sales will continue during the next 24 months at about the same historical rates, and that delivery will also continue to follow the following pace.

Sales of new cars/month	Frequency
6	3
7	4
8	6
9	12
10	9
11	1
12	1

Mr. Thompson's current policy is to order 14 cars at a time (2 full truckloads, with 7 autos on each truck) and to place a new order whenever the stock on hand reaches 12 autos. What are the results of this policy when simulated over the next 2 years?

Answer:

The average demand is 8.75

The average lead time is 1.86

The average end inventory is 6.56

The average cost sales is 4.04

The daily demand for books at AA Bookshop for the past 200 days is shown in the table below.

Demand for books	0	1	2	3	4	5
Frequency	10	20	40	60	40	30

UNIT 8

LINEAR PROGRAMMING: SIMPLEX METHOD

OUTLINE

Session 8.1: Introduction

Session 8.2: Standard form of LP problems

Session 8.3: Solving LP problems

Session 8.4: Interpreting the Simplex Tableau

OBJECTIVES

By the end of the unit**,** you should be able to:

1. Identify and formulate Linear Programming Problems.
2. Solve Linear Programming Problems using the simplex method.
3. Identify and explain the status resources from the optimum simplex tableau.
4. Identify and explain dual (shadow) prices from the optimum simplex tableau.

Note: In order to achieve these objectives, you need to spend a minimum of four (4) hours and a maximum of six (6) hours working through the sessions.

SESSION 8.1: INTRODUCTION

The simplex method is the general procedure for solving LP problems. It was developed by George Dantzig in 1947.The simplex method is an algorithm.

An algorithm is a set of rules or a systematic procedure for finding the solution to a problem. It s simply a process where a systematic procedure is repeated (iterated) over and over again until a desired result is obtained.

The Simplex algorithm is a method (or computational procedure) for determining basic feasible solutions to a system of equations and testing the solutions for optimality.

SESSION 8.2: STANDARD FORM OF LP

LP includes constraints of all types(≥, ≤, =). The variables may be nonnegative or unrestricted in sign.

To develop a general solution method, the LP problem must be put in a common format, which we will call the standard form. The properties of this form are as follows:

i. All the constraints are equations.
ii. All the variables are nonnegative.
iii. The objective function may be maximization or minimization.

An LP model can now be put in the standard form as follows:

1. Constraints:

i. A constraint of the type ≤ (≥) can be converted to an equation by adding a slack variable to(subtracting a surplus variable from) the left side of the constraint.

For example, in the constraint

$X_1 + 2X_2 \leq 6$

We add a slack variable $S_1 \geq 0$ to the left side to obtain the equation

$X_1 + 2X_2 + S_1 = 6$

Now consider the constraint

$3X_1 + 2X_2 - X_3 \geq 6$

Since the left side is not smaller than the right side, subtract a surplus variable $S_2 \geq 0$ from the left hand side to obtain the equation

$3X_1 + 2X_2 - X_3 - S_2 = 6$

ii. The right side of an equation can always be made nonnegative by multiplying both sides of the equation by –1(negative one).

For example,

$3X_1 + 2X_2 - X_3 = -6$ is mathematically equivalent to

$-3X_1 - 2X_2 + X_3 = 6$

iii. The direction of an inequality is reversed when both sides are multiplied by –1 (negative one)

For example, whereas $2<4$, $-2> -4$. Thus, the inequality

$5X_1 + 2X_2 - X_3 \geq -6$ can be replaced by

$-5X_1 - 2X_2 + X_3 \leq 6$

SESSION 8.3: SOLVING LP PROBLEMS

The following example will be used to demonstrate the use of the Simplex method.

Example 1:

Maximize: $Z = 5X_1 + 3X_2$

Subject to:

$6X_1 + 2X_2 \leq 36$

$5X_1 + 5X_2 \leq 40$

$2X_1 + 4X_2 \leq 28$

$X_1, X_2 \geq 0$

Procedure:

1. Initialization step:

a. Express the objective function in the standard form and convert the inequalities to equations by adding slack variables.

$$Z - 5X_1 - 3X_2 = 0$$

$$6X_1 + 2X_2 + S_1 = 36$$

$$5X_1 + 5X_2 + S_2 = 40$$

$$2X_1 + 4X_2 + S_3 = 28$$

b. Express the constraint equations in matrix form

$$\begin{bmatrix} 6 & 2 & 1 & 0 & 0 \\ 5 & 5 & 0 & 1 & 0 \\ 2 & 4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 40 \\ 28 \end{bmatrix}$$

c. Set up an initial Simplex Tableau composed of the coefficient matrix of the constraint equations and the column vector of the constant set above a row of the indicators which are the coefficients of the objective function in the standard form and a zero coefficient for each slack variable.

Basic Variables	Equation/ Row	Coefficient of						Constant
		Z	X_1	X_2	S_1	S_2		
Z	1	1	-5	-3	0	0	0	0
S_1	2	0	6	2	1	0	0	36
S_2	.3	0	5	5	0	1	0	40
S_3	4	0	2	4	0	0	1	28

d. Read the first initial basic feasible solution. Select the original variables to be the initial non-basic variables (set equal to zero) and the slack variables to be the initial basic variables.

That is: $X_1 = 0$ and $X_2 = 0$. Hence $S_1 = 36$, $S_2 = 40$ and $S_3 = 28$

2. Iteration step:

a. Determine the entering basic variable by selecting a variable with the negative coefficient having the largest absolute value. That column becomes the pivot column. In this case, the number is –5, and X_1 becomes the pivot column.

b. Determine the leaving basic variable using the minimum ratio test. This is done by using each coefficient in the pivot column to divide the elements of the constants. The row with the smallest ratio (called pivot row) determines the variable to leave the basis. The process of selecting the variable to be included and the variable to be excluded is called change of basis.. In the current example, $\frac{36}{6}$ is the smallest ratio $\left(\frac{36}{6} \prec \frac{40}{5} \prec \frac{28}{2}\right)$. So row one(1) is the pivot row. Since the unit vector with one(1) in the first row appears under S_1, S_1 leaves the basis.

c. Determine the new basic feasible solution by pivoting.. Pivoting involves converting the pivot element to one(1) and all the other elements in the pivot column to zero(0).The pivot number/element is the number at the intersection of the column of the variable entering the basis(i.e. the element at the intersection of the pivot row and the pivot column) In this case the number is 6 This is done using Gaussian elimination method. As follows:

   i. Multiply the pivot row by the reciprocal of the pivot element. In this case, multiply by $\frac{1}{6}$.

Basic Variables	Equation/ Row	Coefficient of Z	X_1	X_2	S_1	S_2	S_3	Constant
Z	1	1	-5	-3	0	0	0	0
S_1	2	0	1	$\frac{1}{3}$	$\frac{1}{6}$	0	0	6
S_2	.3	0	5	5	0	1	0	40
S_3	4	0	2	4	0	0	1	28

   ii. Having reduced the pivot element to one(1), clear the pivot column as follows:

Add 5 times row 2 to row 1

Subtract 5 times row 2 from row 3

Subtract 2 times row 1 from row 4

Basic Variables	Equation/ Row	Coefficient of						Constant
		Z	X_1	X_2	S_1	S_2	S_3	
Z	1	1	0	$-\frac{4}{3}$	$\frac{5}{6}$	0	0	30
S_1	2	0	1	$\frac{1}{3}$	$\frac{1}{6}$	0	0	6
S_2	.3	0	0	$\frac{10}{3}$	$-\frac{5}{6}$	1	0	10
S_3	4	0	0	$\frac{10}{3}$	$-\frac{1}{3}$	0	1	16

The second feasible solution can be read from the second tableau by setting X_2 =0 and S_1 =0 and we will be left with an identity matrix which gives X_1 =6, S_2 = 10 and S_3 = 16

3. <u>Optimization step</u>:

The current basic feasible solution is optimal if and only if every coefficient in equation 1(the objective function) is non-negative. If it is then stop, otherwise go to the iteration step to obtain the next basic feasible solution

Since there is a negative coefficient in equation 1 (the objective function), we continue the iteration. The only negative indicator is $-\frac{4}{3}$ in the second column. So X_2 is introduced into the basis column 2 becomes the pivot column. Dividing the constants' column by the pivot column shows that the smallest ratio is in the third row. Thus, $\frac{10}{3}$ becomes the new pinot element (i.e. the number at the intersection of the pivot row and pivot column)

Converting the pivot element (10/3) to one and all the other elements in the pivot column to zero(0) as follows:

i. Multiply the pivot row by the reciprocal of the pivot element. In this case, multiply by $\frac{3}{10}$

Basic Variables	Equation/ Row	Coefficient of						Constant
		Z	X_1	X_2	S_1	S_2	S_3	
Z	1	1	0	-4/3	5/6	0	0	30
S_1	2	0	1	1/3	1/6	0	0	6
S_2	.3	0	0	1	-1/4	3/10	0	3
S_3	4	0	0	10/3	-1/3	0	1	16

ii. Having reduced the pivot element to one(1), clear the pivot column as follows:

Add 4/3 times row 3 to row 1

Subtract 1-3 times row 3 from row 2

Subtract 10/3 times row 3 from row 4

Basic Variables	Equation/ Row	Coefficient of						Constant
		Z	X_1	X_2	S_1	S_2	S_3	
Z	1	1	0	-4/3	1/2	2/5	0	34
S_1	2	0	1	1/3	1/4	-3/10	0	5
S_2	.3	0	0	1	1/4	3/10	0	3
S_3	4	0	0	10/3	1/2	-1	1	6

The third feasible solution can be read from the tableau. Setting $S_1 = 0$, $S_2 = 0$, we have an identity matrix which gives $X_1 = 5$, $X_2 = 3$ and $S_3 = 6$

Since there are no negatives indicators left in the first row(objective function), this is the optimal solution. The maximum value of Z is read from the constant column. In this case, it is 34.

There are no slacks in the two constraints, indicating that the first two inputs are all used up. However, 6 units of the third input remain unused.

EXAMPLE 2:

Maximize: $Z = 3X_E+2X_I$

Subject to:

$X_E+2X_I \leq 6$

$2X_E+2X_I \leq 8$

$-X_E+X_I \leq 1$

$X_I \leq 2$

$X_E, X_I \geq 0$

Procedure:

1. Initialization step:

i. Express the objective function in the standard form and convert the inequalities to equations by adding slack variables.

$Z - 3X_E-2X_I + 0S_1+ 0S_2+ 0S_3+ 0S_4=0$

$X_E+2X_I + S_1 +0S_2+ 0S_3+ 0S_4 =6$

$2X_E+2X_I+ 0S_1 + S_2 + 0S_3+ 0S_4 = 8$

$-X_E+X_I + 0S_1 + 0S_2+ S_3+ 0S_4 =1$

$X_I + 0S_1+ 0S_2+ 0S_3+ S_4 = 2$

ii. Express the constraint equations in matrix form

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_E \\ x_I \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \\ 2 \end{bmatrix}$$

iii. Set up an initial Simplex Tableau composed of the coefficient matrix of the constraint equations and the column vector of the constant set above a row of the indicators which are the coefficients of the objective function in the standard form and a zero coefficient for each slack variable.

Basic Variables	Equation/ Row	Coefficient of							Constant
		Z	X_E	X_I	S_1	S_2	S_3	S_4	
Z	1	1	–3	–2	0	0	0	0	0
S_1	2	0	1	2	1	0	0	0	6
S_2	3	0	2	1	0	1	0	0	8
S_3	4	0	–1	1	0	0	1	0	1
S_4	5		0	1	0	0	0	1	2

iv. Read the first initial basic feasible solution. Select the original variables to be the initial non-basic variables (set equal to zero) and the slack variables to be the initial basic variables.

That is: $X_E = 0$ and $X_I = 0$. Hence $S_1 = 6$, $S_2 = 8$, $S_3 = 1$ and $S_4 = 2$

2. <u>Iteration step:</u>

i. Determine the entering basic variable by selecting a variable with the negative coefficient having the largest absolute value. That column becomes the pivot column. In this case, the number is –3, and X_E becomes the pivot column.

ii. Determine the leaving basic variable using the minimum ratio test. This is done by using each coefficient in the pivot column to divide the elements of the constants. The row with the smallest ratio (called pivot row) determines the variable to leave the basis. The process of selecting the variable to be included and the variable to be excluded is called change of basis. In the current example, 8/2 is the smallest ratio

(8/2< 6/1). So row one(1) is the pivot row. Since the unit vector with one(1) in the first row appears under $S_{1,}$ S_1 leaves the basis.

iii. Determine the new basic feasible solution by pivoting.. Pivoting involves converting the pivot element to one (1) and all the other elements in the pivot column to zero (0). The pivot number/element is the number at the intersection of the column of the variable entering the basis (i.e. the element at the intersection of the pivot row and the pivot column) In this case the number is 2 This is done using Gaussian elimination method. As follows:

i. Multiply the pivot row by the reciprocal of the pivot element. In this case, multiply by 1/2.

Basic Variables	Equation/ Row	Coefficient of							Constant
		Z	X_E	X_I	S_1	S_2	S_3	S_4	
Z	1	1	–3	–2	0	0	0	0	0
S_1	2	0	1	2	1	0	0	0	6
S_2	3	0	1	½	0	½	0	0	8/2 = 4
S_3	4	0	–1	1	0	0	1	0	1
S_4	5		0	1	0	0	0	1	2

ii. Having reduced the pivot element to one(1), clear the pivot column as follows:

Add 3 times row 3 to row 1

Subtract row 3 from row 2

Add row 3 to row 4

Maintain row 5

Basic Variables	Equation/ Row	Coefficient of							Constant
		Z	X_E	X_I	S_1	S_2	S_3	S_4	
Z	1	1	0	–½	0	3/2	0	0	12
S_1	2	0	0	3/2	1	–½	0	0	2
S_2	.3	0	1	½	0	½	0	0	4
S_3	4	0	0	3/2	0	½	1	0	5
S_4	5		0	1	0	0	0	1	2

The second feasible solution can be read from the second tableau by setting X_I =0 and S_2 =0 and we will be left with an identity matrix which gives X_E =4, S_1 = 2 S_3 = 5 and S_4 = 2.

3. <u>Optimization step</u>:

The current basic feasible solution is optimal if and only if every coefficient in equation 1(the objective function) is non-negative. If it is then stop, otherwise go to the iteration step to obtain the next basic feasible solution

Since there is a negative coefficient in equation 1 (the objective function), we continue the iteration. The only negative indicator is –½. So X_I is introduced into the basis and it becomes the pivot column. Dividing the constants' column by the pivot column shows that the smallest ratio is in the second row. Thus, 3/2 becomes the new pivot element (i.e. the number at the intersection of the pivot row and pivot column)

Converting the pivot element (3/2) to one and all the other elements in the pivot column to zero (0) as follows:

i. Multiply the pivot row by the reciprocal of the pivot element. In this case, multiply by 2/3

Basic Variables	Equation/ Row	Coefficient of							Constant
		Z	X_E	X_I	S_1	S_2	S_3	S_4	
Z	1	1	0	–½	0	3/2	0	0	12
S_1	2	0	0	1	2/3	–1/3	0	0	2/(3/2)=4/3
S_2	.3	0	1	½	0	½	0	0	4/ (½)=8
S_3	4	0	0	3/2	0	½	1	0	5/(3/2)=10/3
S_4	5		0	1	0	0	0	1	2/1=2

ii. Having reduced the pivot element to one(1), clear the pivot column as follows:

Add ½ times row 2 to row 1

Subtract ½ times row 2 from row 3

Subtract 3 times row 2 from row 4

Subtract row 2 from row 5

Basic Variables	Equation/ Row	Coefficient of							Constant
		Z	X_E	X_I	S_1	S_2	S_3	S_4	
Z	1	1	0	0	1/3	4/3	0	0	12 1/3
S_1	2	0	0	1	2/3	–1/3	0	0	4/3
S_2	.3	0	1	0	–1/3	2/3	0	0	10/3
S_3	4	0	0	0	–1	1	1	0	3
S_4	5		0	0	–2/3	1/3	0	1	2/3

The third feasible solution can be read from the tableau. Setting $S_1 = 0$, $S_2 = 0$, we have an identity matrix which gives $X_E = 10/3$, $X_I = 4/3$, $S_3 = 3$ and $S_4 = 2/3$

Since there are no negatives indicators left in the first row(objective function), this is the optimal solution. The maximum value of Z is read from the constant column. In this case, it is 12 2/3

SESSION 8.4: INTERPRETING THE SIMPLEX TABLEAU

The following information can be obtained from the simplex tableau.

i. The optimum solution.

ii. The status of the resources

iii. The dual prices and reduced costs.

Illustration

Maximize $Z=3X_1+2X_2$

Subject to: $X_1+2X_2 \leq 6$(raw material A)

$2X_1+2X_2 \leq 8$(raw material B)

$X_1+X_2 \leq 1$(raw material C)

$X_1 \leq 2$(raw material D)

$X_1, X_2 \geq 0$

The optimal tableau is shown below:

Basic Variables	Equation/ Row	Coefficient of							Constant
		Z	X_1	X_2	S_1	S_2	S_3	S_4	
Z	1	0	0	0	1/3	4/3	0	0	12 1/3
S_1	2	0	0	1	2/3	–1/3	0	0	4/3
S_2	.3	1	1	0	–1/3	2/3	0	0	10/3
S_3	4	0	0	0	–1	1	1	0	3
S_4	5	0	0	0	2/3	1/3	0	1	2/3

From the table, we have the following:

<u>**8.4.1: Optimal solution**</u>:

Decision variable	Optimum value	Decision
X_1	10/3	Produce 10/3 tons of X_1
X_2	4/3	Produce 4/3 tons of X_2
Z	12 1/3	Result is 12 1/3 thousand dollars

8.4.2: Status of the resources:

A constraint is classified as scarce or abundant depending respectively on whether or not the optimum solution consumes (uses) the entire available amount of the associated resources. This can be deduced from the final(optimal) simplex tableau by observing the values of the slack variables.

i. A positive slack means that the resource is used completely, which means it is abundant

ii. A zero slack indicates that the entire amount of resource is consumed (used) by the activities of the model.

For the above example, we have the following:

RESOURCE	SLACK	STATUS
Raw material A	$S_1 = 0$	Scarce
Raw material B	$S_2 = 0$	Scarce
Raw material C	$S_3 = 3$	Abundant
Raw material D	$S_4 = 2/3$	Abundant

8.4.3: The dual price

These are read from the equation 1 as follows:

$S^*_1 = 1/3$

$S^*_2 = 4/3$

$S^*_3 = 0$

$S^*_4 = 0$