Module 4

Junior Secondary Mathematics

Algebraic Processes

The Commonwealth of Learning

Science, Technology and Mathematics Modules

for Upper Primary and Junior Secondary School Teachers of Science, Technology and Mathematics by Distance in the Southern African Development Community (SADC)

Developed by
The Southern African Development Community (SADC)

Ministries of Education in:

- **Botswana**
- **Malawi**
- **Mozambique**
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- **South Africa**
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SCIENCE, TECHNOLOGY AND MATHEMATICS MODULES

This module is one of a series prepared under the auspices of the participating Southern African Development Community (SADC) and The Commonwealth of Learning as part of the Training of Upper Primary and Junior Secondary Science, Technology and Mathematics Teachers in Africa by Distance. These modules enable teachers to enhance their professional skills through distance and open learning. Many individuals and groups have been involved in writing and producing these modules. We trust that they will benefit not only the teachers who use them, but also, ultimately, their students and the communities and nations in which they live.

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Module 3: *Shapes and Sizes*
Module 4: *Algebraic Processes*
Module 5: *Solving Equations*
Module 6: *Data Handling*

A MESSAGE FROM THE COMMONWEALTH OF LEARNING

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Dato' Professor Gajaraj Dhanarajan
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Introduction

Welcome to *Algebraic Processes*, Module 4 of Teaching Junior Secondary Mathematics! This series of six modules is designed to help you to strengthen your knowledge of mathematics topics and to acquire more instructional strategies for teaching mathematics in the classroom.

The guiding principles of these modules are to help make the connection between theoretical maths and the use of the maths; to apply instructional theory to practice in the classroom situation; and to support you, as you in turn help your students to apply mathematics theory to practical classroom work.

Programme Goals

This programme is designed to help you to:

- strengthen your understanding of mathematics topics
- expand the range of instructional strategies that you can use in the mathematics classroom

Programme Objectives

By the time you have completed this programme, you should be able to:

- develop and present lessons on the nature of the mathematics process, with an emphasis on where each type of mathematics is used outside of the classroom
- guide students as they work in teams on practical projects in mathematics, and help them to work effectively as a member of a group
- use questioning and explanation strategies to help students learn new concepts and to support students in their problem solving activities
- guide students in the use of investigative strategies on particular projects, and thus to show them how mathematical tools are used
- guide students as they prepare their portfolios about their project activities

How to work on this programme

As is indicated in the programme goals and objectives, the programme provides for you to participate actively in each module by applying instructional strategies when exploring mathematics with your students and by reflecting on that experience. In other words, you "put on your student uniform" for the time you work on this course.

Working as a student

If you completed Module 1…did you in fact complete it? That is, did you actually do the various Assignments by yourself or with your students? Did you write down your answers, then compare them with the answers at the back of the module?

It is possible to simply read these modules and gain some insight from doing so. But you gain far more, and your teaching practice has a much better chance of improving, if you consider these modules as a *course of study* like the courses you studied in school. That means engaging in the material—solving the sample problems, preparing lesson plans when asked to and trying them with your students, and so on.

To be a better teacher, first be a better student!

Working on your own

You may be the only teacher of mathematics topics in your school, or you may choose to work on your own so you can accommodate this programme within your schedule. Module 1 included some strategies for that situation, such as:

1. Establish a regular schedule for working on the module.

2. Choose a study space where you can work quietly without interruption.

3. Identify someone whose interests are relevant to mathematics (for example, a science teacher in your school) with whom you can discuss the module and some of your ideas about teaching mathematics. Even the most independent learner benefits from good dialogue with others: it helps us to formulate our ideas—or as one learner commented, "How do I know what I'm thinking until I hear what I have to say?"

It is hoped that you have your schedule established, and have also conversed with a colleague about this course on a few occasions already. As you work through Module 4, please continue!

Resources available to you

Although these modules can be completed without referring to additional resource materials, your experience and that of your students can be enriched if you use other resources as well. There is a list of resource materials for each module provided at the end of the module.

ICONS

Throughout each module, you will find the following icons or symbols that alert you to a change in activity within the module.

Read the following explanations to discover what each icon prompts you to do.

	Introduction	Rationale or overview for this part of the course.
	Learning Objectives	What you should be able to do after completing this module or unit.
	Text or Reading Material	Course content for you to study.
	Important—Take Note!	Something to study carefully.
	Self-Marking Exercise	An exercise to demonstrate your own grasp of the content.
	Individual Activity	An exercise or project for you to try by yourself and demonstrate your own grasp of the content.
	Classroom Activity	An exercise or project for you to do with or assign to your students.
	Reflection	A question or project for yourself—for deeper understanding of this concept, or of your use of it when teaching.
	Summary	
	Unit or Module Assignment	Exercise to assess your understanding of all the unit or module topics.
	Suggested Answers to Activities	
	Time	Suggested hours to allow for completing a unit or any learning task.
	Glossary	Definitions of terms used in this module.

CONTENTS

Module 4: Algebraic processes

Module 4
Algebraic processes

Introduction to the module

This module deals with algebra. Algebra deals with generalised numbers (variables) and studies the relationships among these changing quantities. Algebra can be a useful tool to describe and model real life situations. It requires a certain level of abstract thinking and can be introduced to learners at the time they start to move from the concrete thinking level to a more abstract level of thinking (14 – 16 years). If algebra is introduced too early or at a time when the pupil does not really need it to answer the questions he/she is working on, it might become simply manipulation of letters without meaning.

Aim of the module

The module aims at:

(a) justifying the inclusion of algebra in the mathematics curriculum

(b) practising pupil-centred teaching methods

(c) raising your appreciation for use of games and investigations in pupils' learning of mathematics

(d) reflection on your present practice in the teaching of algebra

(e) making you aware of a variety of learning aids that can be used in the teaching of algebra

(f) extending your knowledge on expansion and factorisation

Structure of the module

This module first looks into the purpose of algebra at the lower secondary level (Unit 1) before moving to the topics traditionally covered: Expansion (Unit 2) and factorisation (Unit 3). The emphasis is on looking at activities for the classroom to make algebra more meaningful and understandable for pupils. A pupil centred approach is encouraged throughout the module by looking at activities that can be set to pupils to involve them making sense of abstract concepts.

The final unit (Unit 4) covers content to extend your knowledge in expansion (binomial expansion) and factorisation of polynomials (factor theorem). Unit 4 content is normally not part of Junior Secondary mathematics teaching, and is presented without the emphasis of classroom activities or practical meaning.

The module does not yet make use of the current available technology: symbolic-manipulation algebraic calculators. This new technology is presently changing dramatically the study of algebra in the western world: from symbolic paper-and-pencil manipulation towards conceptual understanding, symbol sense and mathematical modelling of real life problems.

Objectives of the module

When you have completed this module you should be able to create with confidence, as your own understanding and knowledge has been enhanced, a learning environment for your pupils in which they can:

(i) acquire, with understanding, the knowledge of expansion and factorisation of algebraic expressions

(ii) use paper models to illustrate identities

(iii) use algebraic tiles to model expansion and factorisation

(iv) investigate algebraic fallacies

Prerequisite knowledge

This module assumes you have covered Module 1, especially the unit on sequences and finding the general rule for the *n*th term in a sequence.

Unit 1: Learning and teaching algebra

Introduction to Unit 1

The traditional, formal approach to teaching algebra is to look at algebra as a purely mathematical discipline with no emphasis on relating algebra to day to day situations. Pupils have ‘done algebra’ without really seeing a need for it, resulting in numerous common errors. The current trend is to place algebra, and mathematics in general, in a context. The emphasis is on doing mathematics, recognising connections and modelling real life situations. Algebra is a powerful tool in the hands of the learners provided they understand the uses and the limitations of the tool.

Purpose of Unit 1

In this unit you will look at what algebra is and the stages in the historical development of algebra. You also learn about pupils’ main problems in the learning of algebra. This is important, because being aware of pupils’ problems might guide you to look for ways to present the content to pupils in a way so as to avoid the common errors. You will also reflect on the reasons for including algebra as a topic in the learning of mathematics at the lower secondary level.

Objectives

When you have completed this unit you should be able to:

- explain what algebra is
- state the three main stages in the historical development of algebra
- justify the inclusion of algebra in the mathematics curriculum
- list and exemplify pupils’ main problems in the learning of algebra
- justify the use of manipulatives and models in the learning of algebra

Time

To study this unit will take you about 12 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 1: Learning and teaching algebra

Section A1: What is algebra?

The word algebra is from Arabic origin, it is part of the title of a book written by Mohammed ibn-Musa al-Khwarizmi (approximately 830 AD).

The book's title was 'Hisab al-jabr w'al-muqabalah'. Al-jabr means something like 'completion' or 'restoration' and refers to the transposition of subtracted terms to the other side of an equation. Muquabalah refers to 'reduction' or 'balancing,' that is cancelling out like terms on opposite sides of the equal sign in equations.

Al-Khwarizmi's work is based on Hindu work (Brahmagupta, approximately 600) and works from ancient Mesopotamian origin.

Algebra deals with generalised numbers (variable quantities) and studies the relationships (functions) among these changing quantities. It is sometimes referred to as 'generalised arithmetic,' as rules and patterns in arithmetic can lead to more general algebraic expressions. Algebra is the study of general expressions and general results. In arithmetic one might need to compute $(12 + 45)^2$; in algebra one explores $(a + b)^2$. Algebra allows one to express in a short form relationships that might otherwise need long sentences, as for example the Pythagorean theorem.

Section A2: Historical development of algebra

The symbol system of algebra as we know it today has developed over the ages. Three main stages can be distinguished:

I. The time before Diophantus (ca 250): the **rhetoric stage**.

Initially ordinary language was used to describe the solution to problems. No symbols or special designed signs were used. Everything was described in ordinary language.

II. The time from Diophantus (ca 250) to Viète (ca 1600): the **abbreviation** or **syncopated stage**.

Diophantus was the first known person to start using symbolism (abbreviated words) and is frequently referred to as the father of algebra. (History does not mention a mother!) Diophantus is still remembered today for the "Diophantine Equations": any equation in (usually) several unknowns, whose solutions are required to be integers, for example to find all pairs of integers (x, y) that satisfy
(i) $2x – 3y = 4$ or $x^2 + y^2 = 25$.

In this period mathematicians started to use abbreviations for unknown (but specific) quantities. The main concern during this period was the solving of equations, i.e., finding the value of the unknown quantity represented by a letter(s).

For example in $n + 5 = 13$, with n representing a specific unknown, solving the equations means finding the value of that unknown quantity.

III. From Viète (1540 - 1603, also known as Vieta) to today: the **abstract stage**.

As from the time of Viète, letters were used not only to represent unknown quantities (as in the previous period) but letters were also used to represent **given** quantities (used as parameters). This made it possible, among others, to give general solutions to problems and to generalise patterns.

For example the solution to an equation can be $x = 3t + 4$, the nth term of a sequence can be stated as $u_n = 2n - 3$ for n a natural number, the general form of the equation of a line can be stated as $ax + by + c = 0$, here a, b, and c are parameters (variables to distinguish special cases of the general expression).

Unit 1, Assignment 1

1. Write a quarter page biographical data on each of the following mathematicians:

   (i) Mohammed ibn-Musa al-Khwarizmi

   (ii) Diophantus

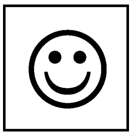
   (iii) Viète

   Information can be extracted from books on the History of Mathematics. If you do not have access to such books you can find some material taken from books in appendices 1, 2 and 3.
2. Using examples illustrate the three major stages in the historical development of algebra.
3. Justify why you would include 'History of Mathematics' in the teaching/learning of mathematics at lower secondary level.

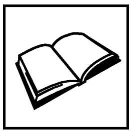
Present your assignment to your supervisor or study group for discussion.

Section A3: Aims of learning algebra

The topics appearing in our syllabi are often taken for granted. We do not question them, we just try to make the best of it by trying to find activities for our students to cover the topic. With algebra, more than with other topics in the syllabus, the questions "Why should pupils at the lower secondary level learn about algebra? What is the use to them of being able to expand, factorise, substitute, solve equations?" are to be faced. The pupils are asking us as teachers. What is our response?



1. What do you consider the main reasons to include algebra in the learning/teaching of mathematics?
2. Read the rational and general aims of the mathematics syllabus you use in your school. Are the reasons you mentioned in 1 in line with the rational and general aims stated in the syllabus?
3. List the content topics you consider core knowledge for pupils aged 12 - 15 and justify each topic. (Why should a pupil have knowledge about the topic you listed?). In which form would you cover the topics you listed?
4. Compare the list you made in 3 with the topics the syllabus wants you to cover with the pupils in each form level. Comment on differences (you might feel that certain topics mentioned in the syllabus should be left out or covered at another level or you might feel that a topic you consider core knowledge is not included, etc).
5. Go through the algebra topics in the mathematics book you use in your school in the classes 1, 2 and 3. How practical are the topics? How useful to pupils in the world of work?
6. A pupil questions why he/she should learn expansion and factorisation of algebraic expressions. What answer would you give the pupil?



When reading the following section refer to the notes you made in your above 'reflection'. See whether your own thinking is confirmed or challenged by the following. If the statements below do not agree with your own reflections critically look at both.

The mathematics syllabus for the forms 1 - 3 of the Junior Secondary School in Botswana emphasises that the learning of mathematics should be

"geared towards developing qualities and skills needed for the world of work"

How practical and useful is school algebra in the world of work and day to day situations?

Most traditional mathematics books present algebra as nothing but manipulation with 'letters'. The algebra presented is not realistic, practical and applicable. If the pupils cannot relate to A + B, and $(a + b)(c + d)$, what practical situation relates to such expressions? Where do they meet $a(c + d)$, apart from in a maths book? Little to no explicit use of algebra is required in the majority of jobs.

Formulas, using sometimes single letter variables, but more often expressed in words, are widely used by technicians, craftsmen, clerical workers and nurses.

For example:

(height of adult in cm) = (mass in kg) + 100;

(amount to pay in P) = (fixed rate in P) + (number of units used × unit price in P);

Child's dose of medicine = $\frac{\text{Age} \times \text{Adult dose}}{\text{Age} + 12}$

All that is required is the **substitution of numbers in these formulae** and the appropriate use of a calculator in the computation. It is not necessary to transform a formula (change of subject). Any form which is likely to be required will be readily available or can be looked up in a manual.

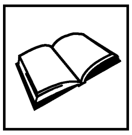
In the world of work and day to day situations it is not necessary to expand expressions (remove brackets), to collect like terms, to factorise (insert brackets) or to solve equations. Yet all these do appear in most if not all syllabi at this level.

It is good to be aware of this: you are to cover content that will be of no use to more than half of the pupils. It is extremely hard to justify—at this level—the value of using letters to represent variables, because pupils do not see any need for it. This is partly due to the fact that the examples used DO NOT need any use of letters.

Consider a question such as: solve for x the equation $x + 2 = 5$. The use of a variable does not make much sense, any primary school child 'knows' what to add to 2 to obtain 5, so why such a complicated way of writing that down?

Similarly in examples on substitution: find the value of $a + b$ if $a = 2$ and $b = 3$; what use do variables have in such cases? You are facing the task to motivate pupils to engage in an activity which has no practical use to them and which is hard to relate to every day work. It is no surprise that, to many pupils, manipulation of letters (which are ill understood to be representing variables or specific unknowns) becomes an isolated practice required to pass an exam.

What are meaningful activities at junior secondary level in algebra?



It might be useful to express an abstract idea or relationship in an algebraic form when this provides a concise way of writing down a structure **which is already recognised** and is **difficult to express neatly in words**. However imposition of an algebraic format or technique which is **not seen to be necessary** and **not related to any familiar situation** cannot serve any useful purpose.

Let's look at examples.

1. In Cash Build customers can buy square tiles (50 cm by 50 cm) to make paths. The manager noted that often a customer will come and say: "I want to make a path two tiles wide around a rectangular region of 3 m by 4 m. How many tiles do I need?" Another customer will come with the

same question but to surround a 3 m by 5 m, rectangular region. A third customer will come with again different dimensions of the region to be surrounded and might want the path to be three tiles wide.

2. On the bank of a river are 8 adults, 2 children and 1 very small rowboat. The owner of the boat charges P0.25 for each crossing and the boat has to end up at the same bank from which it started. The owner does not charge for crossings in which he is involved himself. The group wants to cross the river making as few crossings as possible. The boat can hold 2 children, or 1 child or 1 adult. All people in the group can row. A 'crossing' is moving from one bank to the opposite in either direction. What has the owner to charge for the crossing of the 8 adults and 2 children?

Work the next exercise to find out how these problems lead to algebra. You might want to use a problem solving approach: start with simple cases, tabulate your results and look for a pattern. A systematic approach is needed.

Self mark exercise 1

1. The manager asks you to solve the problem in general. How many square tiles (50 cm × 50 cm) are needed to surround a p m by q m rectangular region with a path of s tiles wide? What relationship between p, q and s would you present to the manager to find the number of tiles needed?
2. The owner of the boat has always counted the number of crossings, but now he has asked you to give him a formula in which he can substitute the number of adults and children in a party to find the amount he is to charge for a party of a adults and c children. What formula would you give to the owner of the boat?

Check your answers at the end of this unit.

What is meaningful for 12 - 15 year old pupils to learn in algebra?

1. Pupils are to become **users of algebra**, not designers.

   Only those aspects of algebra which will be of use to the pupils in their future job, their day to day life or further study (mainly directed towards NON science studies) should be included. Pupils should learn to use algebra as they might meet it in those future situations. Seldom, if ever, will they have to develop new formulas, tables, graphs or algebraic techniques. What is required is to use a given formula, to read a table, to interpret a graph etc.

2. Pupils have to learn to **interpret** tables, graphs, relationships and NOT to MANIPULATE formulas. If manipulation is required it follows from the interpretation, in order to support the interpretation. In the lower secondary school the emphasis should be on interpretation of relationships, graphs and tables as these appear in newspapers, magazines and books.

   If a pupil describes the general term of a pattern as $3n(n-1)+1$ while another pupil in the group comes up with the relation $n^3-(n-1)^3$ the need arises to check whether or not the expressions express the very same thing.

3. Pupils are to study **general techniques** instead of case specific techniques. It is to be preferred that pupils learn a general approach which is applicable in many situations than a narrow technique which is applicable to one situation only.

   For example, there is no need to restrict oneself to linear relationships, for many relationships are nonlinear. Interpretation of aspects of a graph can be practised on various types of graphs not necessarily linear.

   The technique of solving equations by trial and improvement method works for all equations, while techniques for solving linear equations (cover up method, reverse flow method, ‘balance’ method) or quadratic equations (formula, completing the square) are case specific. These special techniques apply to special types of equations only.

   Techniques are to be introduced only when there is a clear need for it. For example when introducing solving of linear equations, using a balance model, and starting with expressions such as $x+7=10$ or $3x+1=22$ does not make much sense as the pupil can ‘see’ the answer immediately. Insisting that pupils write down all the steps

$$x+7=10$$
$$x+7-7=10-7$$
$$x=7$$

   hardly makes sense and pupils don’t see the need of it.

There is no motivation nor justification for learning a new technique unless the pupil meets with a problem which is 'difficult' or impossible to solve **without a new technique**. It is only after it is seen that the techniques available are insufficient to answer the question that the need to look for and introduce other techniques arises.

This is a general principle in the learning of mathematics (and other subjects): before introducing new concepts and techniques a need for it is to be created. The pupils must feel the need to extend their knowledge base with new ideas because a question they like to answer cannot be answered with the knowledge they have. The manager of Cash Build and the rowboat owner (in self mark exercise 1) felt a need for algebraic expressions so they could quickly find the results (number of tiles, price to charge) they wanted.

4. Pupils are to learn to use algebra in order to **describe** in a precise way **relationships** between variables, for example, the relationship between length and mass, height and arm span, number of days after planting and the height of the maize plant, general patterns in arithmetic etc.

   When an algebra course starts with the use of letters as mathematical objects and proceeds to operate on these objects, it remains obscure to pupils what these objects represent and how they can be related (if at all) to realistic situations. Links between arithmetic and algebra are absent.

   The approach since the seventies is to construct algebra from generalised arithmetic (starting with what pupils do know and are familiar with). The first step is using word variables before the move to letter variables is made. Pupils devise and explore situations, recognise and make generalisations from patterns and use algebraic notations. The 'situations' are not to be realistic in the sense that they do occur in a work situation. Pupils, in general, are fascinated by patterns of tiles, matches, circles, dots, cubes, etc. and interested in finding expressions for the general case. The number of patterns one can design is nearly limitless.

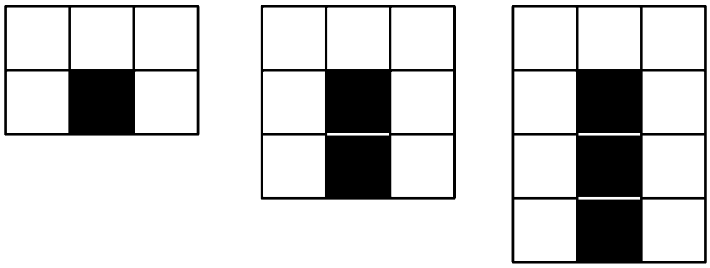
   At all times abstract manipulation of expressions without a context is to be avoided as this will lead only to "fruit salad or cattle post algebra"—the adding of apples and bananas or cows and goats, using letters to label objects instead of using them as variables.

The following exercise illustrates how algebraic expressions follow 'naturally' from an arithmetic pattern of numbers.

Self mark exercise 2

Objective: To generalise from a numerical pattern to an algebraic (word) formula.

1. Here is a pattern of tiles. How many white and black tiles do you need to build the *n*th pattern? If you have N black tiles, how many white tiles do you need?

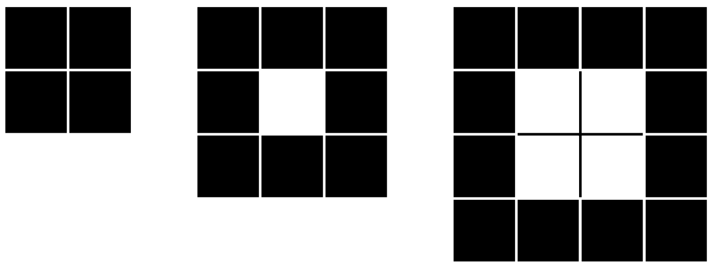


Tabulate your results and generalise:

Pattern	1st	2nd	3rd	4th		*n*th
Number of black tiles	1	2	3	4	...	N
Number of white tiles	5	..	..	..		..

2. Find, in each of the following cases, a generalised expression for the relation between the number of white squares and the number of black squares.

a



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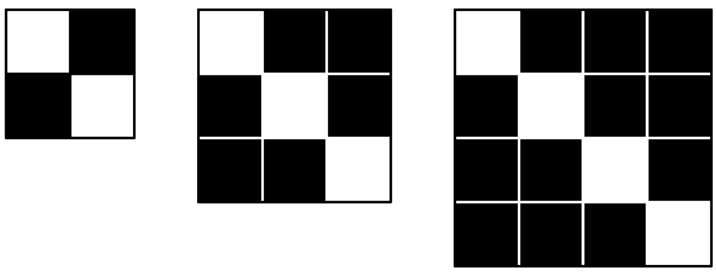


Tabulate your results from simpler cases and look for a pattern.

2a. Pattern	1st	2nd	3rd	4th	...	*n*th
Number of white tiles	0	1	...	...	...	*w*
Number of black tiles	4	...	...	..	...	*b*

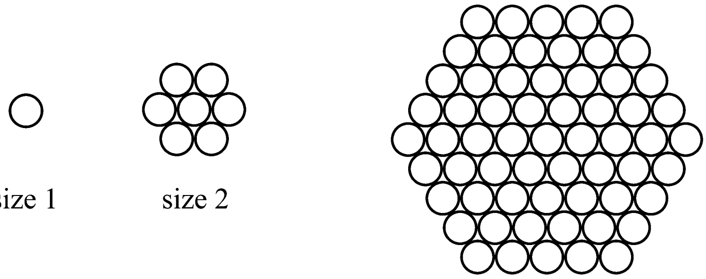
2b. Pattern	1st	2nd	3rd	4th	...	*n*th
Number of white tiles	2	3	..	..	...	*w*
Number of black tiles	2	...	...	...	...	*b*

Self mark exercise 2 continued on next page

Self mark exercise 2 continued

3. A tile pattern is such that (number of white tiles) = 2 × (number of black tiles) – 1. Draw the pattern.
4. When making a cable for a suspension bridge many strands are assembled into a hexagonal formation and then ‘compacted’ together.

   The diagram illustrates a “size” 1, 2 and 5 cable made up of respectively 1, 7 and 61 strands.



size 1 size 2 size 5

   A bridge needs a size 10 cable: How many strands are needed?

   The manager of the cable factory wants to know how many strands she needs to make a cable of any size (let call it size N).

   Tabulate results and look for a pattern as in questions 1 and 2.

Check your answers at the end of this unit.

Unit 1, Practice activity

1. a) Design a worksheet for pupils to investigate the number of tiles needed to make a path around a region. Indicate objective(s) of the activity. To which form would you set the activity?

   b) Try out your worksheet and write an evaluative report.

   Some questions you might want to answer could be: What were the strengths and weaknesses of the activity? What needs improvement? How was the reaction of the pupils? What did you learn as a teacher from the lesson? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils’ investigative abilities? What further activities are you planning to strengthen pupils’ understanding. Were you satisfied with the outcome of the activity?

2. a) Design a worksheet for pupils to investigate the crossing the river problem. Indicate objective(s) of the activity. To which form would you set the activity?

   b) Try out your worksheet and write an evaluative report.

Unit 1, Practice activity continued on next page

Unit 1, Practice activity continued

3. In the text we mentioned four aspects of meaningful learning of algebra for pupils in the junior secondary school age range. What they learn should be

   (i) applicable to situations they are likely to meet in the world of work

   (ii) interpretation of situations expressed in algebraic format (graphs, word formulas)

   (iii) general techniques rather than case specific techniques

   (iv) using algebra to describe a pattern or relationship they are interested in

   Describe clearly, illustrating with examples, the implications for the teaching of algebra of the above i.e. how does it affect the teaching of algebra? Is that reflected in the book you are using in the class? If not, what changes are needed?

4. It is emphasised in the text that pupils will be motivated to learn new concepts and skills when they feel a need for it.

   What problems/questions would you set to pupils to create a need for

   (i) fractions (ii) negative numbers (iii) variables

   (iv) solving linear equations (v) quadratic formula

Present your assignment to your supervisor or study group for discussion.

Section A4 : Pupils' learning difficulties with algebra

Algebra is an ill understood part of mathematics. Although pupils might manage to manipulate with letters this does not imply understanding. The numerous errors made by pupils in algebra point to the fact that ill understood 'rules' are applied to situations in which they are not applicable.

1. Make a list of the common errors you have seen in pupils' work in algebra.

   Try to categorise the errors. That is, place errors you feel have the same root cause in the same category.

2. Write down what you understand by **variable.**
3. Express in algebraic format: there are 4 times as many pupils in the school as there are teachers, using the variables p for the number of pupils and t for the number of teachers.

Now read the following text and compare with your own reflection on the issue.

Research (Hart, 1981) mentions the following as the main difficulties of pupils in the learning of algebra.

I. ***Concept of variable.***

Research agrees that the main difficulty in algebra is the concept of **variable**. The idea that a letter is used to represent ANY NUMBER taken from a certain set (usually N, Z, Q or $\Re$—the notation used is from module 1) is hard for pupils to understand. The distinction between a letter as an unknown (specific) quantity (to be found by solving the equation: $3x^2 + 2x - 8 = 0$) and a letter as a **parameter** (the gradient m and the y-intercept n in $y = mx + n$ are parameters) is also hard for many pupils.

(a) Letters for objects

Letters in algebraic expressions are frequently viewed in a different way by pupils. They might see the **letter to represent an object**. If in a square the measure of the length of the side is indicated with n, pupils see this as denoting the side rather than the measure of the side. This **error of labelling** is extremely common. For example pupils will write: "Let the side be x", "Let the pencil be p", "Let the exercise book be b" etc.—giving labels to objects, instead of introducing a variable.

This misconception is reinforced by several teachers who say, in order to explain $2a + 3a$, 2 apples plus three apples (so called "fruit salad" algebra). But letters are NOT shorthand for objects! $4y$ does not mean that you have four y's (which is not the same as four times the number y). Using a letter as an object, which amounts to reducing the letter's meaning from the abstract variable to something far more concrete and 'real', allows many pupils to arrive at correct answers. However it will break down at some point, particularly in cases in which it is essential to distinguish between the objects themselves and their number.

The classic example is to express in algebraic form the statement: There are 8 times as many pupils in the school as teachers. The frequently given answer is the error response 8p = t (letters as objects), instead of the correct relationship $p = 8t$.

Expressed in word variables: (number of pupils) = 8 × (number of teachers). Here is another example: if the number of days is d, and the number of weeks is w, what is the relation between d and w? Common error: $w = 7d$ (there are seven days in a week), instead of $7w = d$ or expressed using word variables: 7 × (number of weeks) = (number of days).

The idea of seeing letters as labels (truncated words) rather than as a variable might stem from the use of letters l and b in the relation for the area enclosed by a rectangle. l is seen as truncated "length" and b as truncated "breadth", but l and b are representing the measurements i.e. number of length units and NOT the object (the sides).

(b) Different letters represent different numbers

The following questions are from research into pupils' understanding of algebra.

Questions:

1. Is the following statement $a + b = a + c$ true?

   A always B never C sometimes, that is when .....

2. Is the following statement $a + b = a + 2b$ true?

   A always B never C sometimes, that is when .....

3. Find a relation between x, y and z if given that $x + 5 = b$ and $y + z + 5 = b$

The general response of 11-13 years old to the first and second item is NEVER. They reason in question 1: as b and c are different letters they could not be the same number. To respond correctly to this item demands the concept of both b and c running through the whole set of possible numbers. In the second question it is forgotten that zero is also a possible value the variables can take. In the third, they miss the fact that in $a + a$ the variable MUST represent the same number, while in $x + y$, the variables x and y MIGHT be the same or MIGHT be different.

II. *Conjoining*

Initially many pupils have difficulties in understanding that for example $n + 4$ means add 4 to n, and at the same time is representing the result of that addition (four more than n). Pupils are used to expressions such as: 7 + 8, which is equal in value to 15, so the expression (7 + 8) and its value (15) are represented in different ways. Pupils are used to deterministic answers, a practice frequently reinforced by teachers encouraging an answer oriented culture in the classroom.

'Has everybody the correct answer?' 'What is the answer?' are common questions in a classroom. The emphasis is to be on the process: how did you work that?

The difficulty to accept $n + 4$ as the result of the addition of n and 4 leads to conjoining: giving 'one' answer:

$n + 4 = 4n$, $a + a + a + b = 3ab$ or $aaab$, $3a + 4b = 7ab$.

Expressions such as $n + 4$, $a + b$ are not recognised as legitimate 'answers' (a single term answer), the (in this example) + sign being seen as indicating an operation which still needs to be completed.

Pupils have to learn the difference between an operation schema such as for example, 15 – 7 (subtract 7 from 15) and $3a + b$ (add b to $3a$) and the result of such an operation schema (respectively 8 and $3a + b$ meaning b more than $3a$).

Inability to accept lack of 'closure' is also demonstrated in the responses to the question:

Find a relation between x, y and z if given that $x + 5 = b$ and $y + z + 5 = b$ with an incorrect answer $y = z = \frac{1}{2}x$; pupil is not able to accept $y + z$ as an entity.

III. *Difference in convention of notation used in algebra and arithmetic*

(a) If digits are conjoined in arithmetic it represents an implicit addition.
$43 = 40 + 3$;

$2\frac{1}{2} = 2 + \frac{1}{2}$

In algebra $ab \neq a + b$, but means $a \times b$, i.e., an implicit multiplication.

(b) In arithmetic a letter refers to a unit 2 m = 2 metres = 200 cm;

in algebra $2m = 2 \times m$

This leads to errors such as: If $y = 3$ than $4y = ...$; answers given included 7 and 43

IV. *Problems with the grammatical structure of the problem*

Language problems lead to translating an English sentence wrongly into an algebraic equivalent, for example in: 'There are ten times more pupils in the school than teachers', giving the answer $10p = t$.

V. *Common errors in algebra*

When algebra is not related to relational understanding and to realistic models it becomes a 'meaningless' manipulation of letters. "Rules" that apply in one situation are erroneously applied to other situations, i.e. rules are over generalised. The following examples you will recognise and must have come across yourself. You will be able to extend the list!

Correct	Error form
$(x \times y)^2 = x^2 \times y^2$	$(x + y)^2 = x^2 + y^2$ or $\sqrt{x^2 + y^2} = x + y$
$\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$	$\frac{x}{y+z} = \frac{x}{y} + \frac{x}{z}$
$\frac{ab}{ad} = \frac{b}{d}$	$\frac{ab+c}{ad} = \frac{b+c}{d}$

Unit 1, Assignment 3

1. Describe four difficulties pupils have in the learning of algebra. Illustrate each with several examples you met when covering algebra with your classes.
2. Study each of the following pupils' statements. What misconception is reflected in the pupil's answer? Describe in detail the four remedial steps (diagnosing the error, creating conflict, resolving the conflict, consolidation of the correct concept) you would take to assist the pupil to overcome the misconception.

   a. You can't add 2*a* and 3*b* together because it's like 2 apples and 3 bananas.

   b. $2x + 1 = 7$ and $2y + 1 = 7$ are different equations because they have different letters.

   c. If you add 3 onto 4*a* you get 7*a*.

   d. You can't do $p + q = 10$ because there isn't an answer.

   e. (i) In this school there are three times as many girls than boys, so if *b* stands for boys and *g* for girls then in this school it is true that $b = 3g$.

      (ii) ^-x is a negative number.

Present your assignment to your supervisor or study group for discussion.

Summary

Awareness of the "pitfalls of algebra," which were introduced in this unit, is pivotal to your teaching. In the ensuing units, study how the authors seek to make the traditional manipulations relevant to this age group—and thus less prone to common errors—by creating a need, or by using citations from life to make use of algebra realistic.

Unit 1: Answers to self mark exercises

Self mark exercise 1

1. If the number of square tiles is N then $N = 8(s^2 + ps + qs)$.
2. To be paid P0.25 $\times f$, where $f = 4a + 2c$, $c > 2$

Self mark exercise 2

1. Tabulate cases and generalise

Pattern	1st	2nd	3rd	4th		*n*th
Number of black tiles	1	2	3	4	...	N
Number of white tiles	5	7	9	11		2N + 3

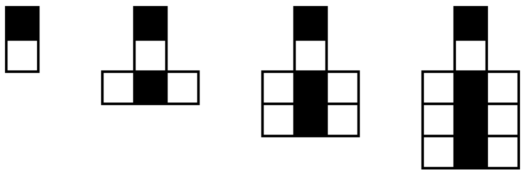
2a.

Pattern	1st	2nd	3rd	4th		*n*th
Number of white tiles	0	1	4	9	...	w
Number of black tiles	4	8	12	40		$4w + 4$

2b.

Pattern	1st	2nd	3rd	4th		*n*th
Number of white tiles	2	3	4	5	...	w
Number of black tiles	2	6	12	20		$w(w - 1)$

3. For example:



4. size 10 needs 271 strands

Size of cable	1	2	3	4	...	N
Strands		1	7	19	37	$3N(N - 1) + 1$

Unit 2: Expansion of expressions

Introduction to Unit 2

Expanding algebraic expression is a traditional topic in algebra. With symbolic-manipulation algebraic calculators available the paper-and-pencil approach will become less and less important in the years to come. The processes involved will get more attention than the actual algorithm. A similar change has taken place in arithmetic: paper-and-pencil long multiplication and division have lost much of their importance. What has become more important is to know when to multiply, when to divide and to use approximations to verify calculator displays. This unit, by providing concrete models pupils can relate to, moves away from the abstract approach.

Purpose of Unit 2

In this unit you will learn about models that can be used in the teaching of expansion of expressions of the format $a(b + c)$, $(ax + by)(cx + dy)$, $(ax + by)^2$. If pupils at this age do not relate abstract ideas to concrete models, algebra becomes pure manipulation of 'letters' without real relational understanding of what is happening and what it can represent.

Objectives

When you have completed this unit you should be able to:

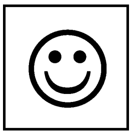
- use the area model in expansion of expressions
- use a multiplication table structure to multiply two polynomials
- use the FOIL acronym in expanding the product of two binomials
- use algebra tiles in the expansion of $(ax + b)(cy + d)$ where a, b, c and d are integers
- illustrate geometrically the identities $(a + b)(a - b) = a^2 - b^2$, $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ using paper models.
- use generalised arithmetical patterns to obtain algebraic identities
- justify the use of manipulatives and models in the learning of algebra
- implement with confidence the models for expansion (area model, multiplication table, FOIL, algebra tiles) in the teaching of algebra
- set investigative activities in which pupils generalise arithmetical patterns and express the generalization algebraically
- use paper models in the classroom to illustrate algebraic identities
- evaluate the effectiveness of using models in the teaching of expansions as compared to a more abstract method

Time

To study this unit will take you about 12 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Section A1: Collecting like terms

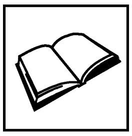
One of the ‘first steps’ in algebra is simplifying expressions. How did you learn about it? What were you told by your teacher? What are you doing with your own pupils?



1. Write down an outline for a lesson in which you want pupils to learn the following:

   collecting of like terms

   Use your outline and compare when working through the following section.
2. Is your outline different from the way you were taught collecting of like terms when at secondary school? Explain and justify similarities and differences.



In setting activities to cover the concept of collecting like terms, you have to keep in mind the basic ideas mentioned in the previous section:

(i) you must avoid pupils seeing ‘letters’ as labels of objects. That letters are representing variables should be clear in the model.

(ii) you must use concrete models to which pupils can relate.

What is clearly to be avoided is ‘fruit salad and cattle post’ algebra. Do not explain $a + a$ as one apple and another apple makes 2 apples so $a + a + 2a$, or $3c + 2g + c + 4g$ as 3 cows and one cow make 4 cows, two goats and four goats makes 6 goats, so $3c + 2g + c + 4g = 4c + 6g$. Although this gives ‘correct’ answers, the conceptual idea (letters representing objects) is invalid and is exactly what you like to avoid taking root into pupils’ minds.

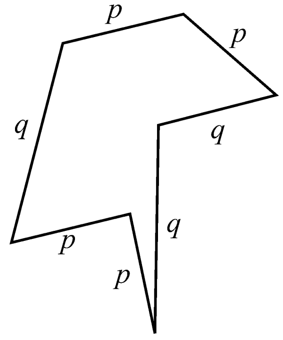
Here are two suggestions.

I. Perimeter model for collecting like terms

You have different rods, some of the same length and others of different lengths. You did not measure them, but you know that some rods have a length of p cm, others of q cm, etc. You have also some rods which you know are 1 cm in length. The rods are used to make polygonal shapes and you find their perimeters in terms of p, q, etc.

Here are three examples illustrated.

Example 1

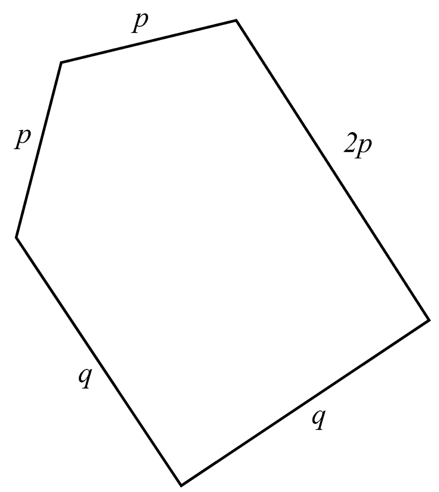


What is the perimeter of this polygon? The lengths of the sides are in cm.

Expected is that pupils will write $p + p + p + q + p + q = 4p + 2q$. Perimeter is $(4p + 2q)$ cm

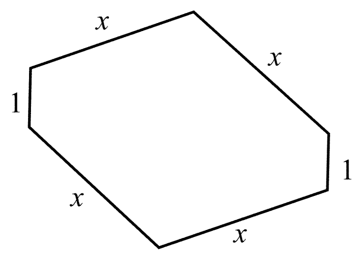
In words: the perimeter is the sum of 4 rods of length p cm and 2 rods of length q cm.

Example 2



Find the perimeter of this polygon. The length of the sides are in cm.

Example 3



Find the perimeter of this polygon. The length of the sides are in cm.

Pupils should also relate given expressions to the perimeter model. Questions such as: represent each of the following expressions using a perimeter model (i) $2a + b$ (ii) $2a + 3$ should be set.

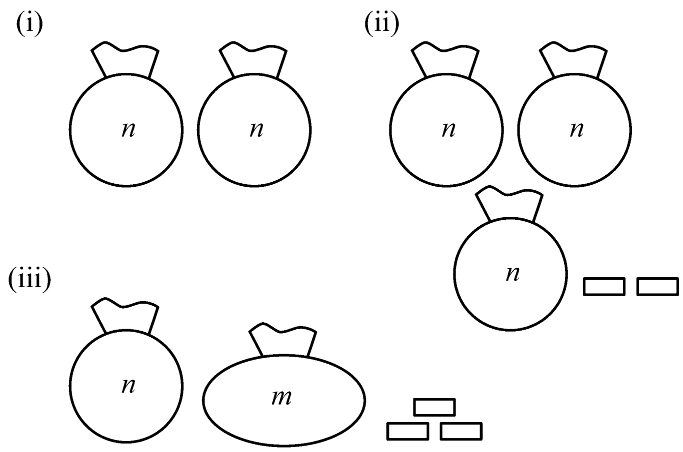
Self mark exercise 1

1. Write down the working you expect a pupil to give in example 2 and example 3.
2. Using the perimeter model represent $2a + b$ and $2a + 3$.

Check your answers at the end of this unit.

II. Bags with coins model

You have bags containing unknown numbers of coins and some single coins (not in the bags). How many coins are there altogether in each of the following cases?



You expect pupils to answer respectively (i) $2n$ (ii) $3n + 2$ (iii) $n + m + 3$ coins.

Unit 2, Practice activity

1. Use the above models for collecting like terms in your classroom. You might have to develop some worksheets for the pupils to work on.

   Write an evaluative report on the activity. Comment on the two models (perimeter/coins in bags). Is one more appealing to pupils than the other?
2. Develop another concrete model to model collecting of like terms and try it out with your pupils.
3. Do the models always work? How would you model $2a - 2$? $3x - 2y$?
4. Consolidation of concepts is always needed. On the following page you find a game like situation for pupils to consolidate the collecting of like terms. Develop it fully and try it out with your class and write an evaluative report. What is the disadvantage of the activity?
5. Look at the outline of the lesson for collecting like terms you wrote at the beginning of this section. Comment on differences (if any) and similarities with the activities suggested above.

Present your assignment to your supervisor or study group for discussion.

Consolidation of collecting like terms: a pupils' activity

Use a square grid placing the designated number of red (r) and yellow (y) counters in each cell. For example:

End

4r	y	3r	3r	8y	r	2y
3y	2r	7y	4y	6r	y	2r
r	4y	5r	5y	2r	2y	4y
5y	6r	2y	r	3y	r	4r

Start

Moving from Start to End from one cell to the other—to the right (R) or up (U)—only a number of possible questions can be looked at:

(i) given a route UURRRURRRUR what number of red and yellow counters do you collect following that route?

(ii) what is the route to be followed if you want the number of red counters to be as small as possible?

what is the route to be followed if you want the number of yellow counters to be as small as possible?

(iii) what is the route to be followed if you want the number of red counters to be as large as possible?

what is the route to be followed if you want the number of yellow counters to be as large as possible?

(iv) which route will give the least TOTAL of counters?

which route will give the maximum number of counters?

Extension:

The up/right movement can be changed to up /down/right/left (diagonal).

Section A2: Expansion of the product of a monomial and a binomial

In this section you are going to look at the expansion of $a(b + c)$, $a(b – c)$ and similar expressions.

1. Write down an outline for a lesson in which you want pupils to learn the following:

   expansion of algebraic products such as $p(x + y)$, $a(b – c)$

   Use your outline and compare when working through the following sections.
2. Is your outline different from the way you were taught expansion when at secondary school?

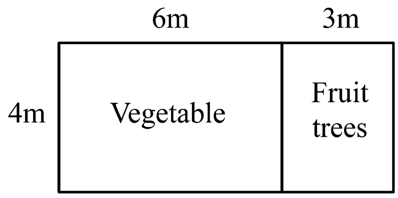
   Explain and justify similarities and differences.
3. Expansion is frequently referred to as "removing brackets". What objections could you bring against the use of this expression?

The area model for expansion

The area model can be used to model expansions. The expansion concept is developed from arithmetic, moving to use of one variable, next to more variables.

Step 1

Pupils were asked to calculate the area of a school garden. The garden is partly used for the growing of vegetables and partly for the growing of fruit trees.



Pupil 1 wrote: $4 \times (6 + 3) = 4 \times 9 = 36\ m^2$

She explained: I multiplied the total length of the garden with the width.

Pupil 2 wrote: $4 \times 6 + 4 \times 3 = 24 + 12 = 36\ m^2$

He explained: I calculated the area of the vegetable garden and the fruit tree garden and added the two together.

They agreed that $4 \times (6 + 3) = 4 \times 6 + 4 \times 3$ (this is called the distributive law: multiplication is distributive over addition).

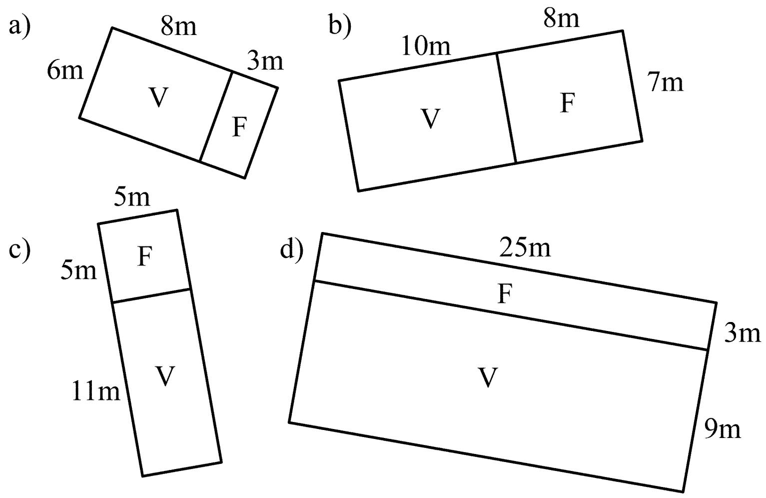
The pupils are set an exercise:

(i) to consolidate the two ways of finding the area of a rectangular garden consisting of two parts (a vegetable and a fruit tree part)

(ii) to draw ‘gardens’ to illustrate a given calculation such as $5 \times (8 + 4)$

Examples of pupil exercises.

1. Calculate the area of each of the following gardens using the method of the first pupil (multiplying the total length of the garden with the width) and also using the method of the second pupil (adding the areas of the two parts).



V Vegetable garden

F Fruit trees

2. Draw vegetable/fruit trees gardens to illustrate these calculations

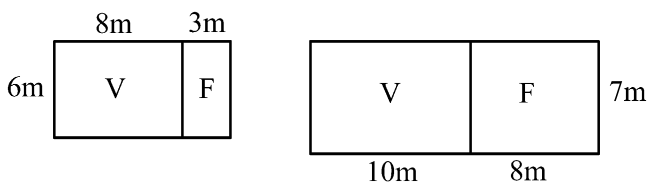
a. $5 \times (9 + 8) = 5 \times 9 + 5 \times 8$

b. $8 \times (7 + 5) = 8 \times 7 + 8 \times 5$

c. $6 \times (6 + 4) = 6 \times 6 + 6 \times 4$

Self mark exercise 2

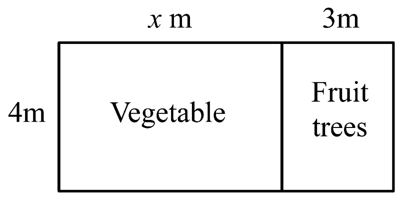
1. Write down the working you expect a pupil to give to question 1 and 2.
2. Why do we have tilted sketches of the gardens and not representations like this?



Check your answers at the end of this unit.

Step 2

The school is to extend the length of the vegetable garden but does not know exactly by how much. The plan is shown in the diagram.



The two pupils use their same method to find the area of the whole garden.

Pupil 1 uses the total length times the width: $4 \times (x + 3) = 4(x + 3)$ m^2.

Pupil 2 adds the area of the vegetable garden to the area of the flower garden:

$4 \times x + 4 \times 3 = (4x + 12)$ m^2

They agree that $4(x + 3) = 4x + 12$.

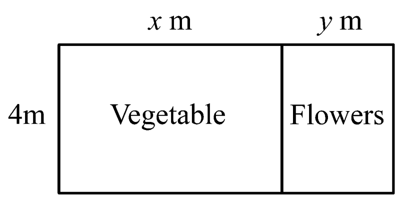
Self mark exercise 3

1. Write an exercise, covering the two types of questions as in step 1, to consolidate the second step.

Check your answers at the end of this unit.

Step 3

A school has a plot to be used for planting vegetables and flowers. They do know the width of the plot but are not yet sure how to subdivide the plot into vegetable part and flower part. They used this plan:



Pupils use the two methods to find an expression for the area of the plot.

Using the total length times the width: $4 \times (x + y) = 4(x + y)$ m^2.

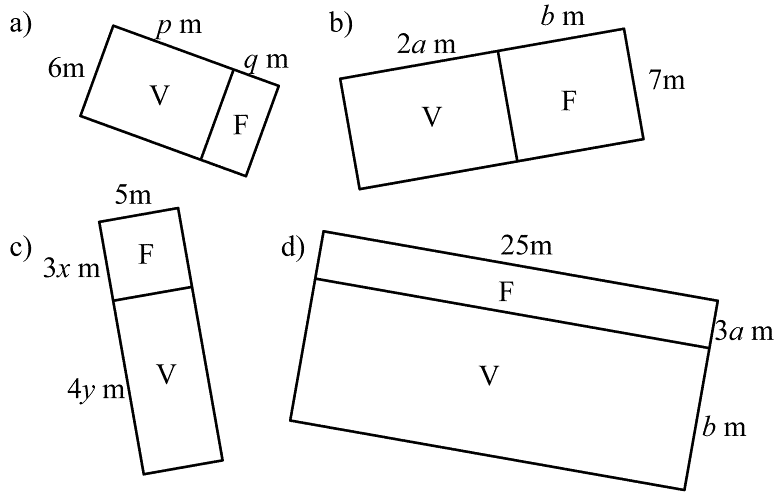
Using addition of the area of the vegetable garden and the area of the flower garden:

$4 \times x + 4 \times y = (4x + 4y)$ m^2

They agree that $4(x + y) = 4x + 4y$.

Examples of questions for the consolidation exercise for the pupils.

1. Calculate the area of each of the following gardens in two ways.



V Vegetable garden

F Fruit trees

2. Draw vegetable/flower gardens to illustrate these calculations

    a) $5 \times (x + 2y) = 5 \times x + 10 \times y$ or $5(x + 2y) = 5x + 10y$

    b) $8 \times (2a + b) = 16 \times a + 8 \times b$ or $8(2a + b) = 16a + 8b$

    c) $6 \times (6x + 4y) = 36 \times x + 24 \times y$ or $6(6x + 4y) = 36x + 24y$

Self mark exercise 4

1. Write down the working you expect a pupil to give to question 1 and 2.
2. Write out the next step, Step 4, to cover the expansion of expressions involving three

    variables such as

    $a(x + y)$, $a(2b + c)$, $4(2x + y + 3)$, $2a(3b + c + 5)$, $2a(a + 3b + 6)$

    Use the last format only if you want to include powers.

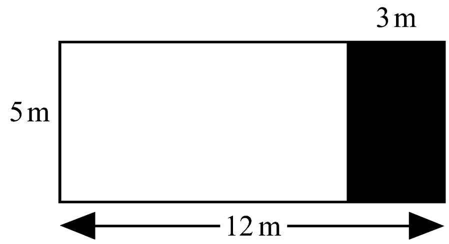
    Make it into a pupil's worksheet.

Check your answers at the end of this unit.

Step 5A

The school is expanding and need more space for buildings. The area available for use as garden is to be reduced.

This is the plan:



To obtain the area of the remaining part the two methods can be applied again:

(i) multiplying the length of the remaining garden section with the width:

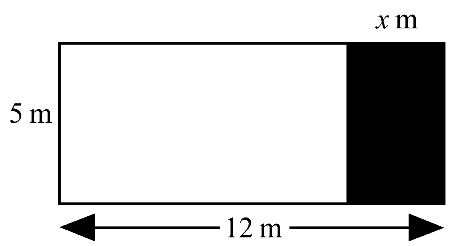
$$5 \times (12 - 3) = 5 \times 9 = 45 \text{ m}^2$$

(ii) subtracting from the original area of the garden the part that is removed:

$$5 \times 12 - 5 \times 3 = 60 - 15 = 45 \text{ m}^2$$

Step 5B

In another school it is not yet known what part is going to be cut. Their plan looks like this:



The two ways of calculating the remaining area of the garden looks now as follows

(i) multiplying the length of the remaining garden section with the width:

$$5 \times (12 - x) = 5(12 - x) \text{ m}^2$$

(ii) subtracting from the original area of the garden the part that is removed:

$$5 \times 12 - 5 \times x = (60 - 5x) \text{ m}^2$$

So that it is concluded that $5(12 - x) = 60 - 5x$

Self mark exercise 5

1. Set an exercise for the pupils to consolidate the above steps 5A & 5B. Remember to cover the two types of questions (expressing given area in two ways, illustrating pairs of equal areas) as in the previous steps.
2. Write out the next steps, step 5C (using two variables) and step 5D (using three variables or more), to cover the expansion of expressions such as 5 $(x - y)$, $a(2b - 4)$, $4x(2y - 3)$ in step 5C and expressions such as $a(3b - c - 5)$, $2a(20 - 3b - 2c)$, $a(4b - c - d)$ in step 5D. Make it into a pupil's worksheet.
3. Can the area model be used to cover and illustrate all possible expansions of a monomial and a polynomial? Justify and illustrate your answer.

Check your answers at the end of this unit.

Multiplication table algorithm for expansion

The area model is a concrete model. A step towards a more abstract approach is using the multiplication table. This is close to the area model, the difference being that two of the sides of the rectangle are deleted. Below are examples in a multiplication table algorithm format.

×	6	3
4	24	12

$4(6 + 3) = 24 + 12$

×	x	3
4	$4x$	12

$4(x + 3) = 4x + 12$

×	a	b
5	$5a$	$5b$

$5(a + b) = 5a + 5b$

×	x	y
a	ax	ay

$a(x + y) = ax + ay$

×	12	$^{-}3$
5	60	$^{-}15$

$5(12 - 3) = 60 - 15$

×	18	^{-}x
10	$4x$	12

$10(18 - x) = 180 - 10x$

×	a	^{-}b
5	$5a$	$^{-}5$

$5(a - b) = 5a - 5b$

×	x	^{-}y
a	ax	^{-}ay

$a(x - y) = ax - ay$

The multiplication table algorithm can be generalised to expressions such as $^-a(b + c)$, $^-a(b - c)$ and $^-a(^-b - c)$ which cannot be interpreted/illustrated with an area model.

$\times$	b	c
^-a	^-ab	^-ac

$^-a(b + c) = {}^-ab - c$

$\times$	b	^-c
^-a	^-ab	ac

$^-a(b - c) = {}^-ab + c$

$\times$	^-b	^-c
^-a	ab	ac

$^-a(^-b - c) = ab + c$

Unit 2, Practice activity

1 Use the above area model for expansion in your classroom. You should have developed worksheets for the pupils to work through the various steps.

Write an evaluative report on the activity.

2. Compare the area model with the multiplication algorithm pointing out advantages and disadvantages of each model. Which would you use in the classroom? Justify.

3 Look at the outline of the lesson for expansion you wrote at the beginning of this section. Comment on differences (if any) and similarities with the activities suggested above.

Present your assignment to your supervisor or study group for discussion.

Section A3: Expansion of the product of two binomials

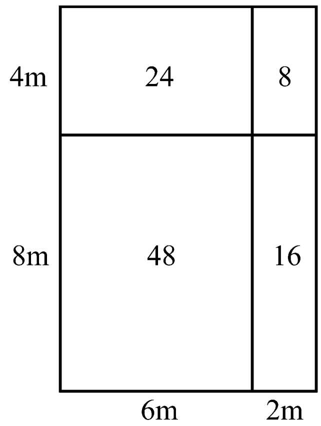
In this section you will extend the work of the previous section (modelling the product of a monomial and a polynomial) to modelling the product of two binomials. The same steps in developing the concepts, using the area model, can be used.

Step 1: Numerical values used only.

A rectangular garden plot is divided into four rectangular parts. The area of the whole garden can be found in two ways.

Method 1: Multiplying the total length of the plot with the total width.

Method 2: Adding the areas of the four parts that make up the total garden.



Area of the garden is
$(4 + 8)(6 + 2) = 12 \times 8 = 96\ m^2$

or

$24 + 8 + 48 + 16 = 96\ m^2$

Hence $(4 + 8)(6 + 2) = 24 + 8 + 48 + 16$

Step 2: One variable is introduced.

This will cover expressions such as $(6 + 3)(a + 4)$.

The first method will give $(6 + 3)(a + 4) = 9(a + 4)$

The second method gives $6a + 24 + 3a + 12$. This simplifies to $9a + 36$

Hence $(6 + 3)(a + 4) = 9(a + 4) = 9a + 36$

Step 3: Two variables are used to cover expressions such as $(a + 3)(b + 6)$.

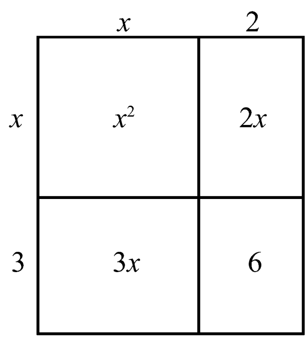
The first method, multiplying total length with total width of the plot, gives the expression
$(a + 3)(b + 6)$.

The second method, adding the areas of the four parts of the garden, will give
$ab + 6a + 3b + 18$.

Hence $(a + 3)(b + 6) = ab + 6a + 3b + 18$.

You might want to include cases with $a = b$. For example:

The school farm is to be divided into four sections. The plan is shown. Next to it is written the two methods that can be used to find the area.



$(x + 2)(x + 3)$

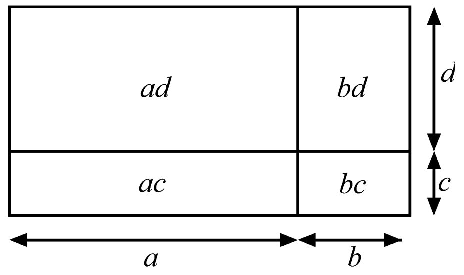
or

$x^2 + 2x + 3x + 6 = x^2 + 5x + 6$

Pupils will agree that $(x + 2)(x + 3) = x^2 + 5x + 6$

Step 4: Extend to the general format $(a + b)(c + d)$.

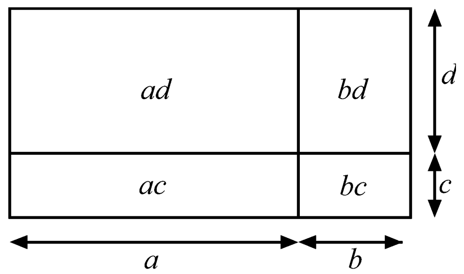
In cases the numbers involved are positive $(a + b)(c + d)$ can be modelled as the area enclosed by a rectangle with sides $(a + b)$ and $(c + d)$.



The area enclosed by the big rectangle equals the sum of the areas enclosed by the four smaller rectangles. Illustrating that
$(a + b)(c + d) = ac + ad + bc + bd$.

In the consolidation exercise for each step you are to cover two types of questions

(i) Given a plot divided into four parts, find two expressions for the total area of the plot. For example a step 3 question could look as illustrated.



(ii) Given an expression, pupils are to illustrate the expression using an area model and use the model to expand the given expression.

Given expression could be for example:

at step 3 level

a $(a + 3)\ (b + 4)$ **b** $(b + 5)(c + 1)$ **c** $(c + 2)(c + 2)$ **d** $(2x + 1)(3y + 4)$

at step 4 level

e $(x + y)(x + y)$ **f** $(2x + 3y)(3x + y)$

Self mark exercise 6

1. Illustrate steps 1 to 4 with area diagrams.
2. Set an exercise, one for each step, for the pupils to consolidate the steps 1 to 4. Remember to cover the two types of questions (expressing given area in two ways, illustrating pairs of equal areas) and to use context (the school garden for example).

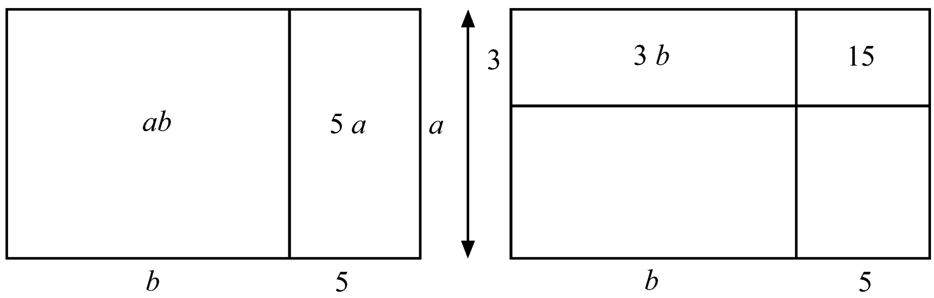
Check your answers at the end of this unit.

In the first four steps all variables involved are (assumed) to be positive real numbers. The area model will work well as long as the variables involved are all positive. In the examples positive whole numbers were used only. However, for higher achievers, including decimals and fractions could be considered.

Expanding $(2.1x + 3.7)(4.8y + 3.6)$ can well be modelled with the area model. The reason to keep the computational work simple is to allow pupils to concentrate on the concept of expansion (use of distributive law) and not to divert their attention by including 'heavy' computational work.

Step 5*:* moving to expressions of the format $(a – b)$ $(p + q)$, $(a – b)(p – q)$. Remember that you modelled in the previous section, expressions such as $a(p – q)$, by cutting part of the school garden.

Let's look at the same idea again. The school reducing the garden by 3 m drew the following plan:



before "cutting"   after cutting $(a – 3)(b + 5)$

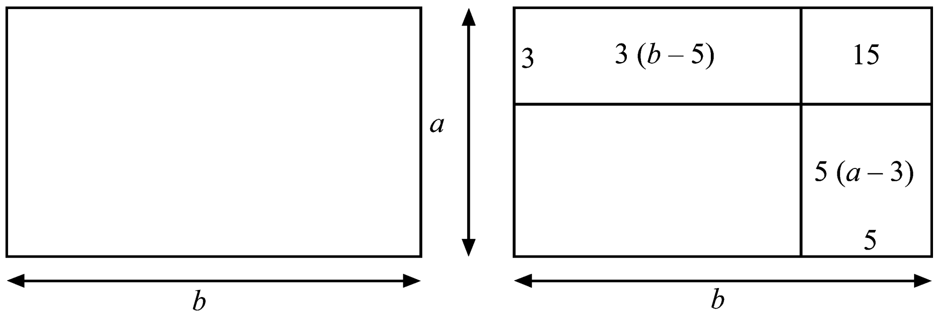
The first method, multiplying length and width of the remaining part of the garden, will give $(a – 3)(b + 5)$.

The second method, subtracting from the original area of $(ab + 5a)$ m^2, the two shaded parts with areas respectively of $3b$ and 15, will give the expression $ab + 5a – 3b – 15$.

The conclusion being that $(a – 3)(b + 5) = ab + 5a – 3b – 15$.

This works well. Now what happens if the school garden is reduced in size by cutting in both directions?

The plan looks like this:



before "cutting" ab after cutting $(a - 3)(b - 5)$

The area remaining of the garden after the cutting is, using the first method, $(a - 3)(b - 5)$.

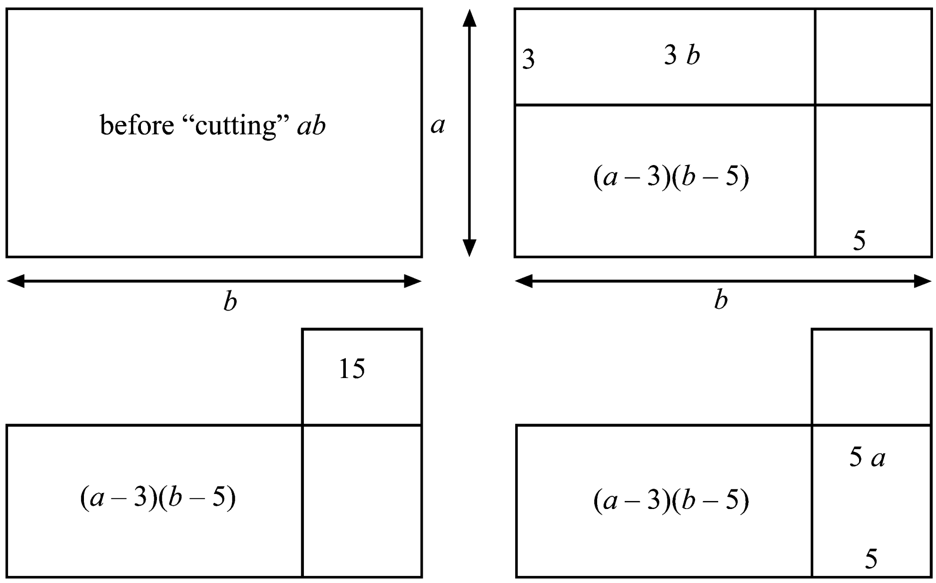
Using the second method the area must also be equal to the original area of ab m^2 minus the areas of the three shaded parts with areas of respectively 15, $3(b - 5) = 3b - 15$ and $5(a - 3) = 5a - 15$.

This leads to the conclusion that
$(a - 3)(b - 5) = ab - (15 + 3b - 15 + 5a - 15) = ab - (3b + 5a - 15)$.

But here we get stuck with our area model as the model does not allow us to interpret $^{-}(3b + 5a - 15)$ as $^{-}3b - 5a + 15$.

Is there a possibility to get around this problem?

Consider the following diagrams.



The first diagram illustrates the initial situation: a garden with area of ab m^2.

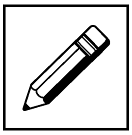
The second diagram illustrates the 'cutting' of an area of 3 m by b m i.e. $3b$ m^2.

In the next diagram 15 m^2 is added in order to make it possible to remove a strip with area $5a$ m^2.

What you have done now is to remove areas of $3b$ and $5a$ and added an area of 15. The remaining area is therefore $ab - 3b - 5a + 15$.

As the two methods give the same area it must be true that

$(a - 3)(b - 5) = ab - 5a - 3b + 15$.



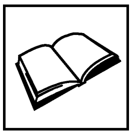
Self mark exercise 7

1. Set an exercise for the pupils to consolidate step 5.

   Remember to cover the two types of questions (expressing given area in two ways, illustrating pairs of equal areas) and to use context (the school garden for example).
2. Have all products of two binomials now been covered? Justify your answer.
3. Set an exercise for higher achievers to extend expansion, using the area model, to expressions such as $(a + b)(p + q + r)$, $(a - b)(p + q + r)$, $(a - b)(p + q - r)$, $(a + b + c)(p + q + r)$, etc.

Check your answers at the end of this unit.

Section B: Models for expansions



The area model discussed above is not the only possible model you can use in the classroom. In this section some more models are considered.

Section B1: Multiplication table algorithm for expansion

The area model fails when negative numbers are involved, as negative lengths of sides of rectangular gardens cannot occur. The area model, being an extremely useful model for expansion, needs some abstraction to allow expansions of expressions such as $(^-a - b)(p - q)$.

Presenting the calculations in a multiplication grid is not so big a step as it remains close to the area model.

Some examples of expansions using the multiplication grid are below.

Example 1: $(p + 3)(p + 2)$

The factors to be multiplied are placed as row and column headings. To expand

$(p + 3)(p + 2)$ the grid looks as shown.

×	p	2
p	$p2$	$2p$
3	$3p$	6

Each variable or number at the top is multiplied by each variable or number at the left, and the results are added : $p\times p + p \times 2 + 3\times p + 3\times 2 = p^2 + 5p + 6$

Example 2: Expansion of $(a + b)(c + d)$ and $(a - b)(^-c + d)$

×	c	d
a	ac	ad
b	bc	bd

$(a + b)(c + d)$
$= ac + ad + bc + bd$

×	$-c$	d
a	^-ac	ad
^-b	bc	^-bd

$(a - b)(c - d)$
$= {^-ac} + ad + bc - bd$

Example 3: Expansion of

1. $(a-2b)(2a+b)$ 2. $(b-3)(2-a)$ 3. $(1-2a)(2a+1)$

$\times$	$2a$	b
a	$2a^2$	ab
${}^-2b$	${}^-4ab$	${}^-2b^2$

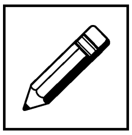
$\times$	2	${}^-a$
b	$2b$	${}^-ab$
${}^-3$	${}^-6$	${}^-3a$

$\times$	$2a$	1
1	$2a$	1
${}^-2a$	${}^-4a^2$	${}^-2a$

$(a-2b)(2a+b)$
$= 2a^2 + ab - 4ab - 2b^2$
$= 2a^2 - 3ab - 2b^2$

$(b-3)(2-a)$
$= 2b - ab - 6 - 3a$
$= {}^-3a - ab + 2b - 6$

$(1-2a)(2a+1)$
$= 2a - 1 - 4a^2 - 2a$
$= {}^-4a^2 - 1$



Self mark exercise 8

1. Use an area model and a multiplication table to expand

   a) $(2a+3)(b+4)$ b) $(b+1)(c-3)$ c) $(2c-3)(3d-1)$
   d) $(d-4)(5-d)$ e) $(5e-11)(5f+11)$ f) $(5-3f)(2-5f)$

2. Use a multiplication table to work $(2a+3)(a^2-2a+3)$.

   Copy and complete:

$\times$	a^2	${}^-2a$	3
$2a$	$2a^3$		$6a$
3			9

   $(2a+3)(a^2-2a+3) = 2a^3$ . . . .

3. Copy and complete the multiplication table for $(a+b)^2 = (a+b)(a+b)$

$\times$	a	b
a		
b		

   $(a+b)^2 =$

Self mark exercise 8 continues on next page

Self mark exercise 8 continued

4. Copy and complete the multiplication table for $(a - b)^2 = (a - b)(a - b)$.

$\times$	a	^-b
a		
^-b		

$(a - b)^2 =$

5. Copy and complete the multiplication table for $(a + b)(a - b)$.

$\times$	a	^-b
a		
b		

$(a + b)(a - b) =$

6. Apply the relationships you obtained in question **3**, **4**, and **5** to expand:

a) $(p + q)^2$ b) $(p - q)^2$ c) $(2p + 3q)^2$

d) $(2p - 3q)^2$ e) $(5p + 6)^2$ f) $(^-p + 2)^2$

g) $(^-4p - 3q)^2$ h) $(4p + 3q)^2$

i) Compare your answer to **g** and **h**.
Will it always be true that $(ap + bq)^2 = (^-ap - bq)^2$? Justify your answer.

j) $(\frac{1}{2}p + \frac{1}{3}q)^2$ k) $(3k + 2)(3k - 2)$ l) $(\frac{1}{3}a + \frac{1}{4}b)(\frac{1}{3}a - \frac{1}{4}b)$

Check your answers at the end of this unit.

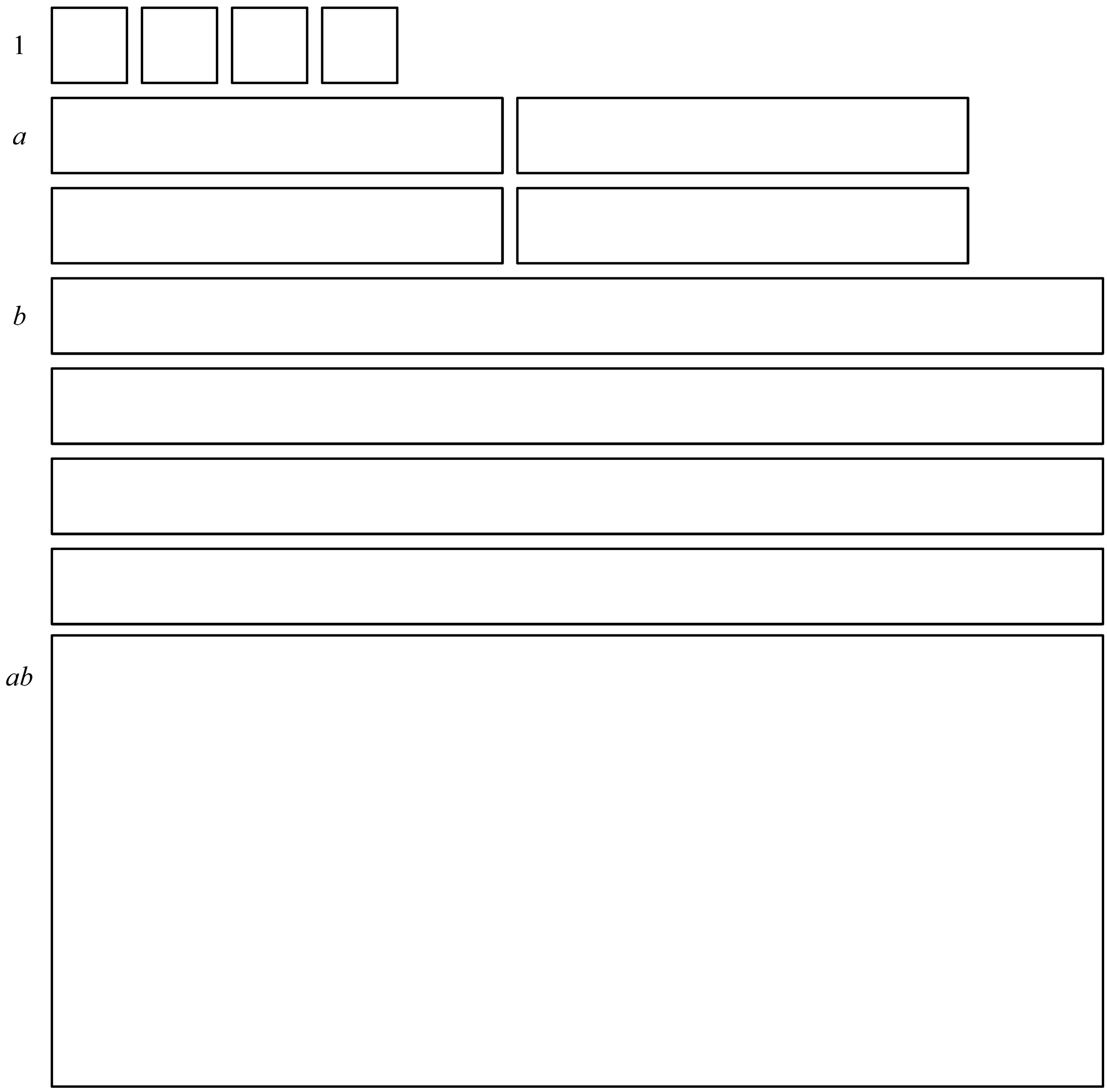
Section B2: Expansion using concrete manipulatives (algebra tiles)

Some pupils understand better when they have concrete objects to manipulate. The objects represent the concepts.

For expansion of expressions of the format
$(a$ + (1st number))$(b$ + (2nd number))
so for example $(a + 2)(b + 3)$, $(a - 2)(b + 3)$, etc.

You need a set of algebra tiles to represent a, b, 1, ab, ^-a, ^-b, ^-ab and $^-1$. You can cut them out, after photocopying them, preferably onto stiffer paper or card stock.

Algebra tiles



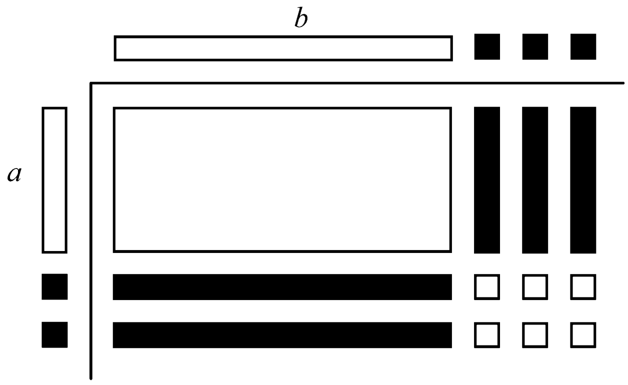

$^{-}1$
^{-}a
^{-}b
^{-}ab

The tiles can be used on an overhead projector for use with the whole class and at the same time pupils can be given individual sets to model expansions with the algebra tiles.

Here is an example of the expansion of $(a - 2)(b - 3)$.

Place tiles representing ***a*** and **$^{-}1$, $^{-}1$** along one side to represent the factor $(a - 2)$.

Along the other side you place tiles ***b*** and **$^{-}1$, $^{-}1$, $^{-}1$** to represent the factor $(b - 3)$.



The region is now filled with the appropriate tiles, keeping in mind the rules for multiplication of directed numbers.

The result gives the following tiles:

ab
$3 \times {}^{-}a = {}^{-}3a$
$2 \times {}^{-}b = {}^{-}2b$
$6 \times 1 \quad = 6$

Adding gives the result of the expansion, thus
$(a - 2)(b - 3) = ab - 3a - 2b + 6$.

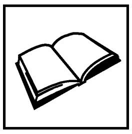
Self mark exercise 9

1. Use your algebra tiles to expand

   a. $(2a - 1)(b + 4)$ b. $({}^{-}a + 2)(2b - 1)$ c. $({}^{-}a - 3)({}^{-}2b + 3)$

2. Compare using algebra tiles with the area model. What are the similarities? What are the differences?

Check your answers at the end of this unit.



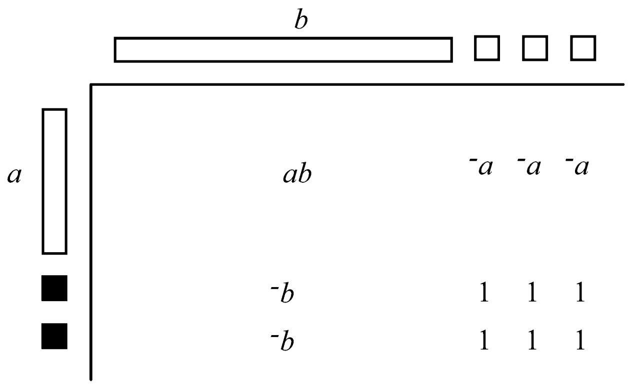
Remember that the area model was abstracted into a multiplication table model. The use of algebra tiles also allows a gradual move to a more abstract representation/algorithm for expansion.

Here are the steps you could consider:

Step 1

Allowing pupils to use the concrete representations as long as they want. A move can be made to a more abstract representation by only placing the factors and writing the tiles 'values'. Some pupils might do this without being prompted!

The diagram illustrates this first step



Step 2

No longer using any concrete manipulative but representing the tiles by their 'values' in a multiplication table:

$\times$	b	$^-3$
a	ab	^-3a
$^-2$	^-2b	6

Section B3: The FOIL algorithm

The algorithm traditionally used for the expansion of the product of two binomials is the FOIL algorithm. FOIL is an acronym for

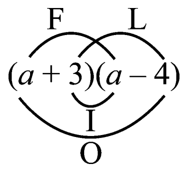
F **F**irst

O **O**uter

I **I**nner

L **L**ast

It looks like this:



This algorithm tells pupils HOW to do it, but the WHY is not clear. This is one of the characteristics of an algorithm: it tells you how you can obtain the required result, but does not explain why the suggested method works.

Many pupils can tell you: "to divide two fractions you multiply the first fraction with the reciprocal of the other," and in algebraic notation they are saying $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. Yet few pupils can explain why this algorithm works.

Unit 2, Practice activity

1. Use the area model and the use of algebra tiles for expansions of the product of two binomials in your classroom. You will have to develop worksheets for the pupils to work through the various steps.

   Compare the two methods.

   Write an evaluative report on the activity. Pay attention to pupils' reaction to use of area model and algebra tiles. What are their preferences?
2. Compare the area model, and the use of algebra tiles with the FOIL algorithm, pointing out advantages and disadvantages of each approach. Which would you use in the classroom? Justify.
3. Explain why the FOIL algorithm works.

Present your assignment to your supervisor or study group for discussion.

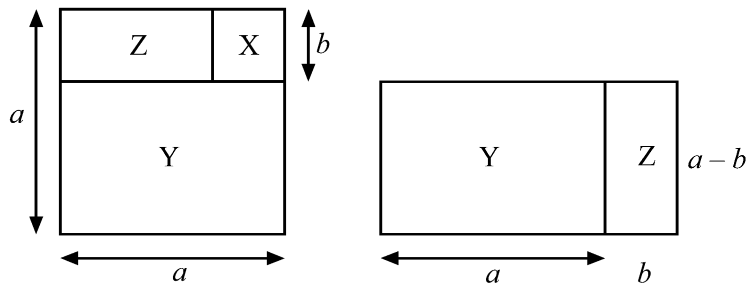
Section B4: Illustrating the identity $(a + b)(a - b) = a^2 - b^2$ with paper models

One way to move away from the abstract nature of algebra is to represent identities in a more concrete way. For several pupils this helps their understanding. Concrete presentation allows pupils to link the abstract idea to something they can see/touch.

The identity $(a + b)(a - b) = a^2 - b^2$.

The diagram below illustrates how pupils should use paste and cut to illustrate the identity $(a + b)(a - b) = a^2 - b^2$.

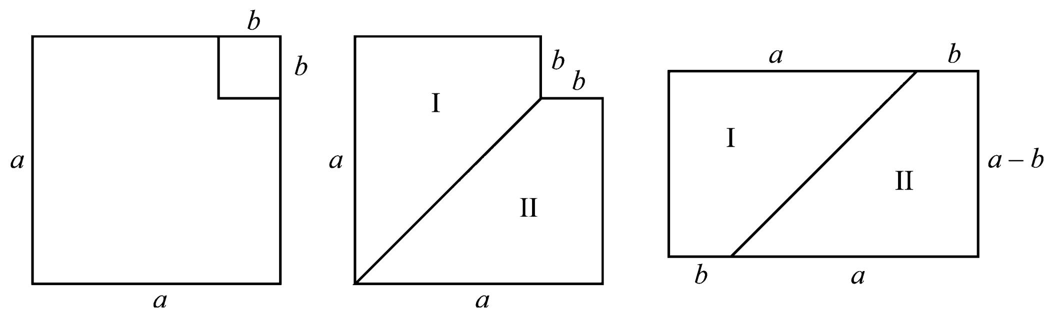
Starting from a large square (side 10 cm for example) mark another square (side 3 cm for example) and also the region Z. Cut out X, Y and Z and rearrange as in second diagram.



The diagram on the left illustrates that the area of the regions Y + Z is equal to the area enclosed by the larger square (with side of length a) minus the area enclosed by the smaller square (with side b). Hence the area of the region Y + Z is numerically equal to $a^2 - b^2$. The diagram on the right is a restructuring of the first diagram. The area of the region Y + Z is now that of a rectangular region with sides of length $(a + b)$ and $(a - b)$ respectively. Hence the area measure is numerically equal to $(a + b)(a - b)$. This illustrates that $(a + b)(a - b) = a^2 - b^2$.

Another way to illustrate the same identity is illustrated in the following diagram. Again it is a cut and paste illustration which can be done by pupils themselves.

From a square (side of length a) a square with a side of length b is removed. The remaining part is cut out (see second diagram), and the two parts I and II are placed to form a rectangle.



Comparing the area enclosed by the first diagram (illustrating $a^2 - b^2$) with the area enclosed by the last diagram (illustrating a rectangle with sides of length $a + b$ and $a - b$) leads to an illustration of the identity

$(a + b)(a - b) = a^2 - b^2$.

For classroom use making a set for the overhead projector might be a useful teaching aid.

Section B5: The product (*a* + *b*)(*a* – *b*) from generalised arithmetical pattern

Arithmetic pattern when generalised can inductively lead to algebraic identities. Work through the following instructions:

Study the following pattern:

$8 \times 8 = 64$	$12 \times 12 = 144$	$104 \times 104 = 10816$
$9 \times 7 = 63$	$13 \times 11 = 143$	$105 \times 103 = 10815$
$10 \times 6 = 60$	$14 \times 10 = 140$	$106 \times 102 = 10812$

...

Add four more rows to the pattern.

Look at the first row.

$8 \times 8 = 64$ $12 \times 12 = 144$ $104 \times 104 = 10816$

What is the general pattern in that row?

Did you recognise the squares?

The pattern in the first row could be written as:

(number) × (number) = $n \times n = n^2$

Compare the numbers used in the second row with the numbers in the first.

$8 \times 8 = 64$	$12 \times 12 = 144$	$104 \times 104 = 10816$	$n \times n = n^2$
↓ ↓ ↓	↓ ↓ ↓	↓ ↓ ↓	↓ ↓ ↓
$9 \times 7 = 63$	$13 \times 11 = 143$	$105 \times 103 = 10815$	

The first 8 ⇒ 9 (add 1) The second 8 ⇒ 7 (subtract 1) and 64 ⇒ 63 (subtract 1)

Check that the same is true for the next expressions in the row.

What has to go with the first n?

What with the second?

What has to go with the n^2?

Did you find $(n + 1)$, $(n - 1)$ and $n^2 - 1$?

The pattern in the second row, when generalised gives

(number + 1) × (number – 1) = (number) × (number) – 1

or in algebraic notation: $(n + 1)(n - 1) = n^2 - 1$

Check that the third row, when compared with the first, will give you the generalisation:

$(n + 2)(n - 2) = n^2 - 4$.

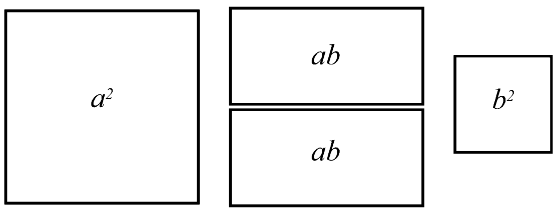
Write down the generalization for the fourth, fifth and *p*th row.

Unit 2, Practice activity

1. Use the paper demonstration of the identity $(a + b)(a - b) = a^2 - b^2$ in your classroom. Write an evaluative report.
2. Try out the pattern method for pupils to generalise to the identity $(a + b)(a - b) = a^2 - b^2$. You will have to adapt the above outline to the pupils' level. In your evaluation compare with the other models you tried out for expansion of the product of two binomials.
3. Develop an worksheet and try it out in the classroom, which when generalised will lead to pupils to discover the identities:

   (i) $(a + b)^2 = a^2 + 2ab + b^2$

   (ii) $(a - b)^2 = a^2 - 2ab + b^2$
4. Make the following tiles

   Use the four pieces to make a square. What is the length of the side of the square? What identity have you illustrated?
5. Expand

   $(a + 1) \qquad = a + 1$

   $(a + 1)^2 = (a + 1)(a + 1) = ...$

   $(a + 1)^3 = (a + 1)(a + 1)(a + 1) = ...$

   $(a + 1)^4 = (a + 1)(a + 1)(a + 1)(a + 1) = ...$

   $(a + 1)^5 = (a + 1)(a + 1)(a + 1)(a + 1)(a + 1) = ...$

   ....

Practice activity continued on next page

Practice activity continued

Describe any pattern you can see.

Find out about Pascal’s Triangle and how it is related to the expansion.

6. Investigate how you could extend the suggested models, algorithms and concrete representations to the expansion of the product of two polynomials. For example to expand $(ax + by + cz)(px + qy + rz)$. Pay attention to ‘short comings’ of some models.
7. How would you encourage able pupils to further investigate expansion? Write an outline for a worksheet on the topic pupils could use to guide their investigation.

Present your assignment to your supervisor or study group for discussion.

Summary

The message of this unit has been clear: all algebraic expansions should be taught from examples of real objects that “behave” that way.

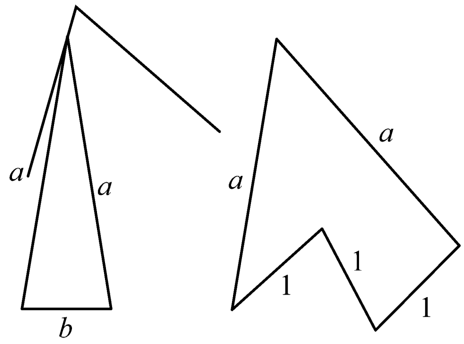
Unit 2: Answers to self mark exercises

Self mark exercise 1

1. $p + p = 2p + q = q = 4p + 2q$

   $x + x + 1 + x + x + 1 = 4x + 2$

2. For example



Self mark exercise 2

1. a. $6 \times (8 + 3) = 6 \times 11 = 66$ m^2

   $6 \times 8 + 6 \times 3 = 48 + 18 = 66$ m^2

   b. $7 \times (10 + 8) = 7 \times 18 = 126$ m^2

   $7 \times 10 + 7 \times 8 = 70 + 56 = 126$ m^2

   c. $5 \times (5+ 11) = 5 \times 16 = 80$ m^2

   $5 \times 5 + 5 \times 11 = 25 + 55 = 80$ m^2

   d. $25 \times (9 + 3) = 25 \times 12 = 300$ m^2

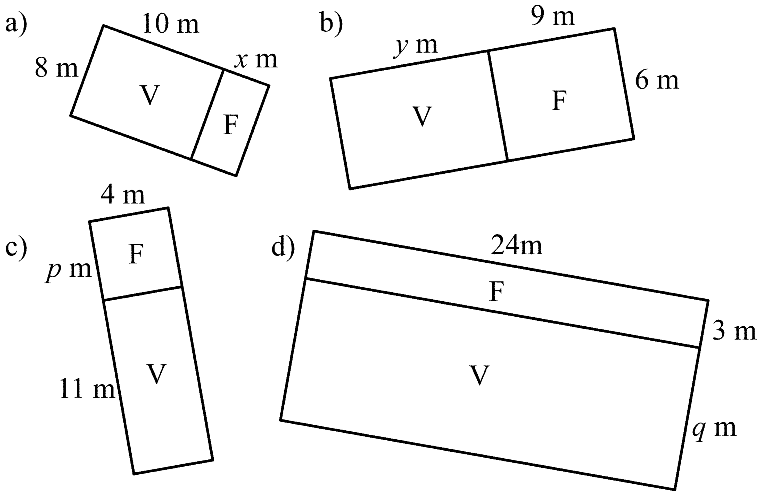
   $25 \times 9 + 25 \times 3 = 225 + 75 = 300$ m^2

2. If pupils ONLY meet representations of rectangles in ‘standard’ position (sides parallel to edges of the page) they might erroneously think that representations of rectangles in tilted positions are NOT rectangles. A multiple format of representation is required to build up the correct concept.

Self mark exercise 3

A few examples are:

1.



V Vegetable garden

F Fruit trees

2. Draw vegetable/fruit trees gardens to illustrate these calculations:

   a) $5 \times (a + 6) = 5 \times a + 5 \times 6$

   b) $7 \times (b + 3) = 7 \times b + 7 \times 3$

   c) $9 \times (6 + c) = 9 \times 6 + 9 \times c$

Self mark exercise 4

1. a) $6(p + q) = 6p + 6q$

   b) $7(2a + b) = 14a + 7b$

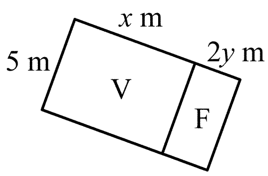
   c) $5(3x + 4y) = 15x + 20y$

   d) $25(3a + b) = 75a + 25b$

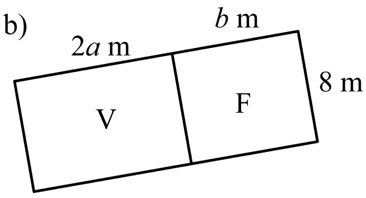
   Or equivalent expressions.

2.

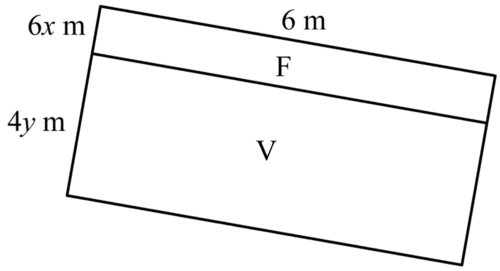
a)



b)



c)

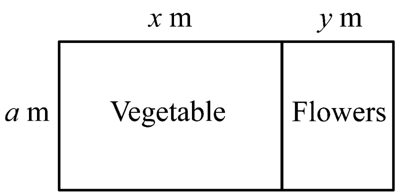


V Vegetable garden

F Fruit trees

2. Outline of Step 4 to be extended into pupils' worksheet.

   A school will get a plot to be used for planting vegetables and flowers. They do not know the width of the plot and are not yet sure how to subdivide the plot into vegetable part and flower part. They used this plan:

   Pupils use the two methods to find an expression for the area of the plot.

   Using the total length times the width: $a \times (x + y) = \mathrm{a}(x + y)$ m^2.

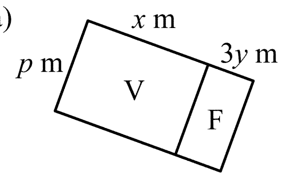
   Using addition of the area of the vegetable garden and the area of the flower garden:

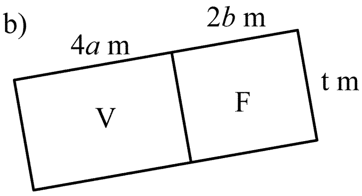
   $a \times x + a \times y = (ax + ay)$ m^2

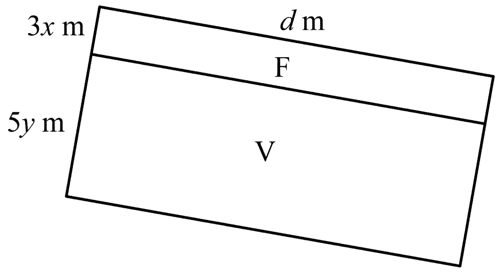
   They agree that $\mathrm{a}(x + y) = ax + ay$.

Examples of questions for the consolidation exercise for the pupils.

1. Calculate the area of each of the following gardens in two ways.

a) 

b) 

c) 

V Vegetable garden

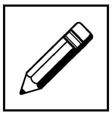
F Fruit trees

2. Draw vegetable/flower gardens to illustrate these calculations:

a) $p \times (x + 2y) = p \times x + p \times y$ or $p(x + 2y) = px + py$

b) $c \times (2a + b) = c \times 2\text{a} + c \times b$ or $c(2a + b) = 2ac + bc$

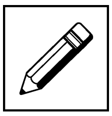
c) $k \times (6x + 4y) = k \times 6x + k \times 4y$ or $k(6x + 4y) = 6kx + 4ky$



Self mark exercise 5

1&2. Similar to previous consolidation exercises.

3. The model will fail in cases such as $^{-}2(4a - 3)$, $^{-}3b(5x - 2y)$ as negative 'length' ($^{-}2$ and $^{-}3b$ respectively, also note that the variable b is assumed to be positive, as otherwise $^{-}3b$ could represent a positive number) cannot be modelled with area. The model also assumes that the variables take positive values i.e. in modelling $a(b + c)$ with an area model a, b, and c are assumed positive.



Self mark exercise 6

Set examples similar to those illustrated in the text. You can find a similar approach in Maths in Action Pupils book 1 and 2 with numerous consolidation exercises.

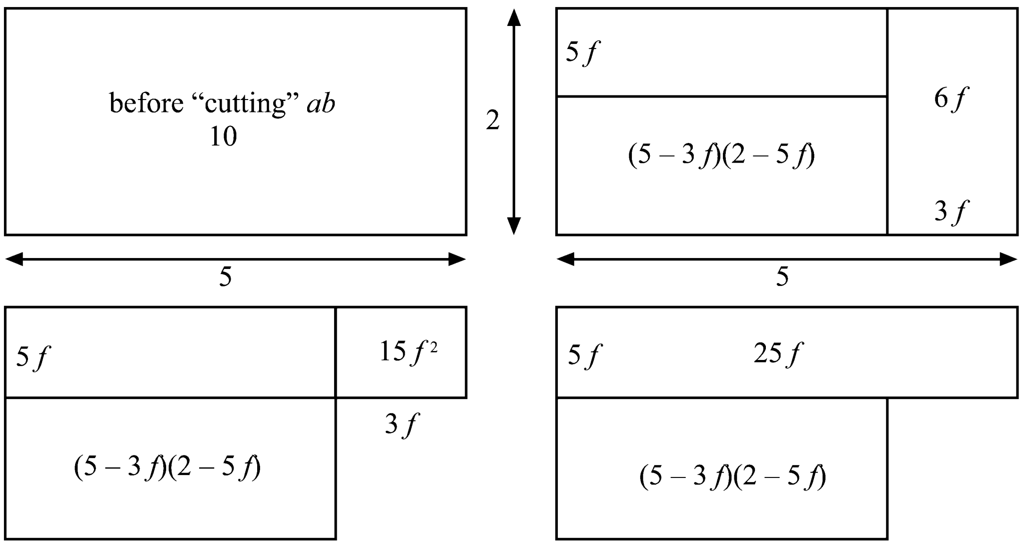
Self mark exercise 7

1 & 3. Set examples similar to those in the text.

2. No! Cases such as ($^{-}p - q$)($a + b$) cannot be represented by an area model.

Self mark exercise 8

1. For example **f** $(5 - 3f)(2 - 5f)$



Start with 5 by 2 rectangle: area 10.
Indicate the strips of width $3f$ and $5f$ to be removed.
First remove a strip of width $3f$ and area $6f$ (second diagram).
The third diagram is the amount remaining.
To allow a strip of $5f$ by 5 to be removed first add $5f$ by $3f$ rectangle (third diagram).
Now the strip with area $25f$ can be removed.
Left is $(5 - 3f)(2 - 5f)$.

This must be equal to the original 10 minus $6f$ plus $15f^2$ minus $25f$

$10 - 6f + 15f^2 - 25f = 10 - 31f + 15f^2$

Hence $(5 - 3f)(2 - 5f) = 10 - 31f + 15f^2$

×	2	$^{-}5f$
5	10	$^{-}25f$
$^{-}3f$	$^{-}6f$	$15f^2$

Hence $(5 - 3f)(2 - 5f) = 10 - 31f + 15f^2$

2.

$\times$	a^2	^-2a	3
$2a$	$2a^3$	$^-4a^2$	$6a$
3	$3a2$	^-6a	9

$(2a + 3)(a^2 - 2a + 3) = 2a^3 - 4a^2 + 6a + 3a^2 - 6a + 9 = 2a^3 - a^2 + 9$

3.

$\times$	a	b
a	a^2	ab
b	ab	b^2

$(a + b)^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

4.

$\times$	a	^-b
a	a^2	^-ab
^-b	^-ab	b^2

$(a - b)^2 = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$

5.

$\times$	a	^-b
a	a^2	^-ab
b	ab	$^-b^2$

$(a + b)(a - b) = a^2 - ab + ab + b^2 = a^2 - b^2$

6. a) $p^2 + 2pq + q^2$ b) $p^2 - 2pq + q^2$ c) $4p^2 + 12pq + 9q^2$
d) $4p^2 - 12pq + 9q^2$ e) $25p^2 + 60p + 36$ f) $p^2 - 4p + 4$
g) $16p^2 + 24pq + 9q^2$ h) $16p^2 + 24pq + 9q^2$
i) $(ap + bq)^2 = (^-ap - bq)^2 = a^2p^2 + 2abpq + b^2q^2$
j) $\frac{1}{4}p^2 + \frac{1}{3}pq + \frac{1}{9}q^2$ k) $9k^2 - 4$ l) $\frac{1}{9}a^2 - \frac{1}{16}b^2$

Self mark exercise 9

1. Straightforward placing of the tiles.
2. Both make use of ‘areas’, but ‘negative’ areas occur with the tile model. The tiles allow us to model all the expansions of expressions of the form $(pa + q)(sb + t)$ where p, q, s and t are integers. The area model did not allow negative values for p and s.

Unit 3: Factorisation of expressions

Introduction to Unit 3

Like expansion (Unit 2), factorisation is not a goal in itself but a tool to be used in, for example, the solution of equations. As the inverse process of expansion of expressions, the methods covered in Unit 2 can be worked backwards to factorise a given expression. Emphasis is on relational understanding by using models.

Purpose of Unit 3

In this unit you will look at the inverse process of expansion of the algebraic expression you studied in Unit 2. Expansion refers to writing a product of factors as the sum of terms. The inverse process is writing the sum of terms as a product. This is called factorisation. You will focus mainly on how the topic can be presented to pupils in your class using concrete models and manipulatives. Your own content knowledge is reviewed, but not extended to, for example, the factor theorem and more general factorisation of polynomials. You might want to look at these topics in any mathematics book for advanced level (form six secondary school) or additional mathematics (forms 4 and 5 secondary school).

In Unit 1 it was mentioned that algebra will rarely be useful for students once they are in the workforce. The examples that introduced expansions in Unit 2—tiles around a fountain, garden areas with vegetables and fruit trees—*suggested* situations where a worker *might* actually find expansions useful. Realistic uses for factorising are by contrast so rare, however, that even that kind of suggestion is not feasible. That is why this unit treats "un-expanding" or checking one's work after expansion, as the practical entry point for teaching factorising.

North American textbooks tend to use the shorter term 'factoring' for factorising.

Objectives

When you have completed this unit you should be able to:

- identify the highest common factor in pairs of expressions
- set activities to pupils to factorise expressions that have a common factor using an area model or a multiplication table structure
- set activities to pupils to discover how to factorise quadratics of the form $x^2 + px + q$
- set activities for pupils to discover the factorisation of the difference between two squares
- use algebraic tiles to factorise quadratics
- use algebra tiles in the classroom with pupils in factorisation
- detect the error(s) in algebraic fallacies

- use fallacies in the teaching of algebra
- justify the use of fallacies in the teaching of mathematics
- use small group discussion as a learning technique

Time

To study this unit will take you about 12 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Section A: Factorising

Section A1: Factorising expressions with one common factor

In this section you are going to look at writing expressions such as $ab + ac$, $ab - ac$, $x^3 + 2x^2 - 4x$ and similar expressions, as a product of two factors. This is the reverse process of expansion.

1. Write down an outline for a lesson in which you want pupils to learn the following:

    factorising expressions such as $ab + ac$, $ab - ac$, $x^3 + 2x^2 - 4x$

    Use your outline and compare when working through the following sections.

    Is your outline different from the way you were taught expansion when at secondary school?

    Explain and justify similarities and differences.

The inverse process of expansion is **factorisation**. This is writing an expression as a product of factors. As factorisation is the reverse process, the models, algorithms and manipulatives used in expansion should be used in 'reverse' order. It also implies that a factorisation can always be checked by expanding the answer. Expanding the answer has to give the original expression. At all times pupils should check answers (not only in this context).

Consider the following example:

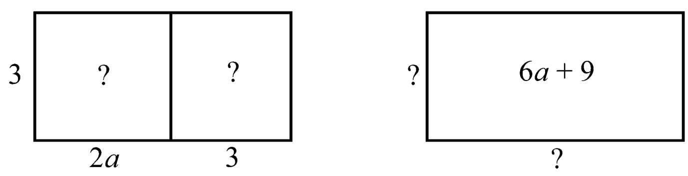
Expansion: $3(2a + 3) = 6a + 9$

Bringing back brackets: $6a + 9 = 3(2a + 3)$,

3 is the highest common factor of $6a$ and 9.

Represented in the area model:

Expansion Factorisation



Expansion: given the sides of the rectangle, find the expression for the area of the rectangle.

Factorisation: given the area enclosed by a rectangle, find the length of the sides.

Self mark exercise 1

1. If the area of a rectangle is 24 cm^2, what are the length and width of the rectangle?
2. If the area of a rectangle is $(6a + 9)$ square units, what are the length and width of the rectangle?

Check your answers at the end of this unit.

Knowing the length of the sides of a rectangle gives one and only one possible answer for the area enclosed by that rectangle.

But knowing the area enclosed by a rectangle you can find infinitely many possible measures for the sides.

How many possibilities did you write down for the rectangle enclosing an area of 24 cm^2? Did you stop because your list is infinitely long?

You might have started with rectangles with dimensions 1 cm × 24 cm, 2 cm × 12 cm, ...etc. Did you include $\frac{1}{2}$ cm × 48 cm, $\frac{1}{3}$ cm × 72 cm, $\sqrt{2}$ cm × 12 $\sqrt{2}$ cm, etc., etc.? In other words there are an infinite number of rectangles enclosing 24 cm^2.

Is the situation any different if the area enclosed is $(6a + 9)$ square units? The answer is NO. Again an infinite number of rectangles will enclose an area of $(6a + 9)$ square units. To list but a few:

$1 \times (6a + 9)$, $2 \times (3a + 4.5)$, $3 \times (2a + 3)$, $1.5 \times (4a + 6)$, $6(a + 1.5)$, $^{-}3(^{-}2a - 3)$, $100(0.06a + 0.09)$, $a \times (6 + \frac{9}{a})$, ...

24 and $6a + 9$ can be expressed as a product of two factors in infinitely many ways. This would make the process of factorisation not very meaningful. However when the word factorisation is used two things are assumed—and pupils should be made aware of these—

(i) factorisation is over a **set of numbers**. This set over which the factorisation is to be carried out is, at secondary school level, assumed to be the (positive) integers.

(ii) factorisation always assumes that the **highest** common factor is used.

Self mark exercise 2

1. Look at the suggested factorisations of $6a + 9$.
   a) Which have to be rejected because the factorisation is over the integers?
   b) Which have to be rejected because you are looking for the highest common factor?
   c) Does $6a + 9$ have a unique factorisation over the integers?
2. Find the highest common factor of each pair of expressions.
   a) $4x$ and 12 b) $24a$ and 9
   c) $15x$ and $60x^2$ d) $4x^2$ and $26x^3$
3. Find the highest common factor of each pair of expressions.
   a) $4x^2$ and $20x$ b) $24a^3$ and $9a^2$
   c) $6x^4$ and $30x^2$ d) $12x^2$ and $16x^3$
4. Find the highest common factor of each pair of expressions.
   a) $20x^2y$ and $12xy^2$ b) $24a^2b^2$ and $9ab^3$
   c) $5xy^3$ and $10x2y^2$ d) $3x2y^4z$ and $27x3y^3z^2$

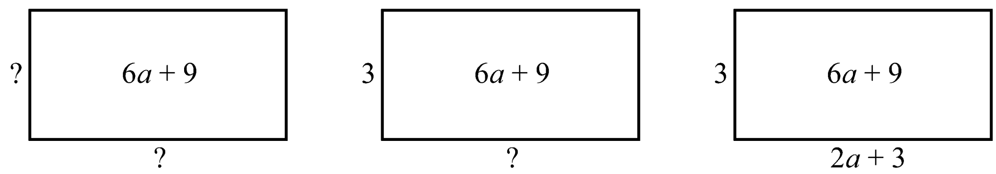
Check your answers at the end of this unit.

Let's return to the area model to represent the factorisation of $6a + 9$. Factorisation means: You are to find the dimensions of the rectangle given its area enclosed.

As the width is to be the highest common factor, the width is 3. (step 2).

Dividing the width into the area: $\frac{6a+9}{3} = 2a + 3$ gives the dimension of the length (step 3).

Factorisation | Factorisation | Factorisation



Step 1: write down the area.

Step 2: Width will be equal to the highest common factor.

Step 3: Divide width into each of the terms of the area to find the length.

Notice that this approach requires that the students be comfortable when handling proper fractions. Some review may be necessary.

To assist pupils in factorisation you go through various stages, starting at numerical level and gradually including variables and factorisation of increasing difficulty level.

Here are some examples of questions you could set to pupils.

Section A2: Outline of a worksheet for pupils on factorisation

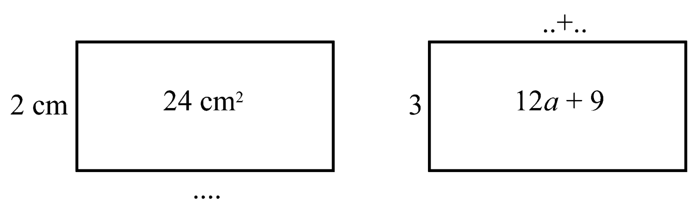
Objective: Pupils should be able to use the area model to factor out a common factor.

It is assumed that pupils have covered an exercise on finding the highest common factor of pairs of expressions.

Worksheet (outline)

Worked example:

Find the length of the missing side of each rectangle.



Now write down the factorisation of the expression.

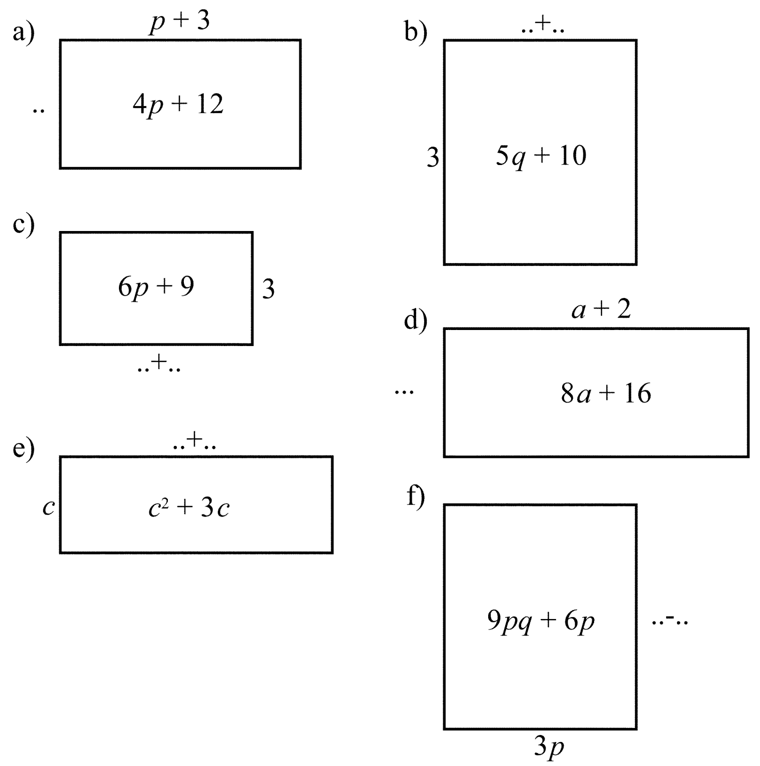
The expected working is: the length of the rectangle is

$24 \div 2 = 12$ cm and $\frac{12a+9}{3} = 4a + 3$ respectively.

Factorisation: $24 = 2 \times 12$ $\quad$ $12a + 9 = 3(4a + 3)$

Question 1

Find the missing side of each rectangle. Write down the expression you factorised.



Question 2

Factorise using the area model. Check your answers using expansion.

a) $3a + 6$ b) $12a + 6b$ c) $4x - 6y$ d) $5x + 10y$ e) $4a^2 + 12b^2$

f) $6a^2 - 2b^2$ g) $8y - 12x$ h) $24a - 38b$ i) $2x + 4y - 6$

Question 3

Factorise using the area model then check by expanding your answer.

a) $3a^2 + 6a$ b) $12b + 6b^2$ c) $4x^2 - 6x$

d) $5x^3 + 10x^2$ e) $4b^2 + 12b$ f) $6a^2 - 2a$

g) $8y^3\text{m} - 12y^2$ h) $4 - 4abc^3$ i) $x^3 + x^2 - x$

Question 4

Factorise using the area model then check by expanding your answer.

a) $3ab^2 + 6ab$ b) $12a^3b + 6a^2b$ c) $4xy^2 - 6y^2$

d) $5x3y^2 + 10x2y$ e) $4a^2b^2 + 12b^2$ f) $6a^2b - 2ab^2$

g) $8xy^3 - 12x2y^3$ h) $4ab - 4abc^3$ i) $2x2y^2 - 6x2y + 8xy^2$

Self mark exercise 3

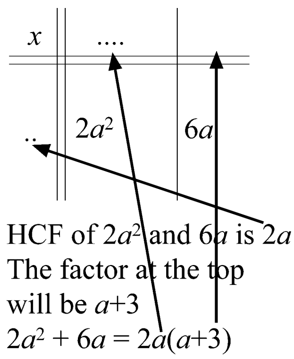
1. Work through the above pupils' worksheet on factorisation.
2. Can any expression, having a common factor, be factorised using the area model? Justify your answer.

Check your answers at the end of this unit.

The area model in factorisation has obviously the same weakness as it had in expansions: negative numbers as factors do not fit in the model. A (slight) step towards abstraction is using the multiplication table structure. In expansion the entries in the first row and column were given, then the body of the table had to be filled in. The expansion of $a(b-3)$ appeared in multiplication table form as illustrated.

×	b	$^-3$
a		

In the case of factorisation: the entries in the cell of the table are known, but the 'headings' are to be found. The factorisation of $2a^2 + 6a$ is illustrated.



A pupils' exercise could include examples similar to the following

Question 1: Complete the multiplication tables and the factorisations:

a)

×	a	. .
. .	$8a$	24

$8a + 24 = . . (a + . .)$

b)

×	. .	. .
3	$12b$	$^-9$

$12b - 9 = 3(. . - . .)$

c)

×	. .	. .
. .	t^2	^-8t

$t^2 - 8t = t(. . - . .)$

Question 2: Use a multiplication table to factorise:

a) $4a + 12$ b) $5p - 30q$ c) $^{-}3d + 9e$

d) $^{-}a^2 - 4a$ e) $^{-}3ab - 6ab^2 + 9a^2$

Self mark exercise 4

1. Work through the pupils' sample questions for an exercise presented above.

Check your answers at the end of this unit.

Unit 3, Practice activity

1. Use the area model and the multiplication table to factorise expressions with a common factor in your classroom. You will have to develop worksheets for the pupils to work through the various steps and levels of difficulty.

   Write an evaluative report on the activity.

2. Develop and try out a worksheet to extend factorisation of expressions having a common term. Justify the structure of your worksheet and the type of questions included.

   Write an evaluative report.

3. Compare the activities suggested in this section with your own lesson outline you wrote at the beginning of this section. Comment on difference and similarities. Would you change your original lesson outline after having gone through this section? Justify your answer.

Present your assignment to your supervisor or study group for discussion.

Section B: Factorising quadratic expressions

In this section you are going to look at various ways in which factorising of quadratic expressions can be presented to pupils. Quadratic expressions are, in general, of the form

$ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

Section B1: Factorising $a^2 - b^2$ using a pattern approach

The following illustrate a possible discovery approach for the factorisation of the difference of two squares. The objective of the activity is that pupils will discover that $a^2 - b^2$ will factorise as $(a + b)(a - b)$. Pupils sit in groups of four, work individually through the worksheet, but can discuss ideas at any moment with each other. If each pupil in the group has completed the worksheet they are to compare, discuss and come up with a conjecture as a group. The first part considers factorisation with a and b being positive integers. The last question **g** extends to non integer values of a and b. Question **h** is a challenge for the pupils and can only be tackled after they have met with square roots.

This is only a first step in the factorisation of the difference of squares: restricted to numerical cases, although leading to a generalised form $a^2 - b^2 = (a + b)(a - b)$.

The next step is to apply the relation to algebraic expressions.

Pupils are to factorise expressions such as

$p^2 - q^2$ $\quad 4p^2 - 9q^2$ $\quad 16x^2 - 36y^2$ $\quad a^2b^2 - 9c^2$ $\quad {}^{-}25a^2 + 64c^2$

And some more challenging expressions such as

$2.25a^2 - 6.25c^2$ $\quad \frac{1}{4}a^2 - \frac{1}{9}b^2$ $\quad \frac{1}{4a^2} - \frac{1}{9d^2}$

Outline of a worksheet on factorisation of $a^2 - b^2$. A group/class discussion.

a) Complete the table:

Difference of two squares	Calculation	Factors	Pattern
$4^2 - 1^2$	$16 - 1 = 15$	3×5	$(4 - 1)(4 + 1)$
$5^2 - 2^2$	$25 - 4 = 21$	3×7	$(5 - 2)(5 + 2)$
$6^2 - 1^2$			
$6^2 - 3^2$			
$7^2 - 4^2$			
$8^2 - 3^2$			
$9^2 - 2^2$			
$10^2 - 3^2$			

b) Can you find a pattern that connects the first column to the last?

c) Check your pattern for

$12^2 - 8^2$ $\quad\quad$ $45^2 - 21^2$

Add these to the table.

d) Check three other differences of two squares and write your results in the table.

e) How can you factorise $a^2 - b^2$?

f) Check your factorisation by working backwards i.e. expanding your expression to see if you get $a^2 - b^2$.

g) Does your expression work for decimals? fractions? Check for example

(i) $(3.2)^2 - (1.3)^2$ $\quad\quad$ (ii) $\left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2$

h) Can you factorise 3 – 2?

Self mark exercise 5

1. Work through the pupil's activity described above to factorise $a^2 - b^2$
2. A pupil said that you can factorise the difference of two squares in more than one way. For example $16 - 25 = 4^2 - 5^2 = (4 - 5)(4 + 5)$
   or also $16 - 25 = (^-4)^2 - 5^2 = (^-4 - 5)(^-4 + 5)$
   or also $16 - 25 = 4^2 - (^-5^2) = (4 - ^-5)(4 + ^-5)$.

   For algebraic forms such as $a^2 - 4b^2$ the pupil gave as factorisation $(^-a + 2b)(^-a - 2b)$ and $(a + 2b)(a - 2b)$.

   How would you respond to this pupil?
3. Factorise completely.

   a) $7^2 - a^2$ b) $b^2 - (0.5)^2$ c) $9c^2 - 16$
   d) $1 - 2.25d^2$ e) $98^2 - 2^2$ f) $54^2 - 46^2$
   g) $(\frac{2}{3})^2 - (\frac{1}{3})^2$ h) $(9.8)^2 - (0.2)^2$ i) $2a^2 - 32$
   j) $5b^2 - 45$
4. Factorise completely.

   a) $(x^2 - \frac{1}{x^2})$ b) $(\frac{4}{9x^2} - \frac{16y^2}{25})$ c) $x^4 - 1$
   d) $(a + b)^2 - (a - b)^2$ e) $(x + 5)^2 - 9$ f) $(3x - 1)^2 - (4x + 5)^2$
   g) $a^4 - b^4$ h) $x^2 - 3$ i) $2x^2 - 5$

Check your answers at the end of this unit.

Unit 3, Practice activity

1. Try out the activities in the pupil's outline worksheet to discover the factorisation of the difference of two squares from an arithmetic pattern. Give sufficient consolidation with numbers before moving to algebraic expressions.

   Write an evaluative report.
2. Develop and try out a worksheet to assist your pupils in the learning of the factorisation of the difference of two algebraic squares. Justify the structure of your worksheet and the type of questions included. The question 3 and 4 in the Self mark exercise 5 might give you some ideas on questions that could be included. You might have to differentiate by content to allow both low and high achievers to meet challenges.

   Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

Section B2: Factorising $x^2 + ax + b$

In this section you are going to look at ways to help pupils to factorise trinomials of the form $x^2 + ax + b$. Two methods are discussed
(i) generalisation from patterns (ii) using algebra tiles

I. Generalizing a pattern to discover the factorisation of $x^2 + ax + b$

As factorisation is the reverse of expansion, pupils are expanding binomials (using a multiplication algorithm) and comparing the expanded form with the original factor product. Reading backwards—from expanded form to factorised form—gives the factorisation of the trinomial. The coefficient of x^2 is kept 1 and is later to be extended.

Pupils sit in groups of four and work individually, but can consult each other at any time. The completed worksheets need comparing and discussion in the group. Some challenges are included at the end of the worksheet to convey to pupils: you cannot always factorise over the integers.

The objective is to have pupils discover that to factorise $x^2 + ax + b$ they are to find two numbers p and q, with sum a and product b. If such numbers can be found the expression factorises as $(x + p)(x + q)$. The multiplication table is used to find the values of p and q.

The factorisation is restricted to factorisation over the integers. The value for p and q to look for are integers. This also indicates the limitation of the method, that very, very few trinomials can be factorised. To impress this on pupils in the challenge activities, one can include some trinomials which do not factorise over the integers or not factorise at all. (Although strictly speaking ALL trinomials can be factorised over C, the set of complex numbers).

Outline of a worksheet on factorisation of $x^2 + ax + b$. A group/class discussion.

a) Expand the expression in the first column using area model or multiplication table model.

×	a	4
a	a^2	$4a$
3	$3a$	12

$7a$

Place the result below and complete the table.

Factorized form **$(a + p)(a + q)$**	***p***	***q***	**Expanded form** **$a2 + sa + t$**	***s***	***t***
$(a + 4)(a + 3)$	4	3	$a^2 + 7a + 12$	7	12
$(a + 1)(a + 2)$	1	2			
$(a + 2)(a + 4)$					
$(a – 2)(a + 5)$					
$(a – 3)(a + 4)$					
$(a – 2)(a – 5)$					

b) Can you find a pattern that connects the values of *s* and *t* to the values of *p* and *q*?

Write it down in words and in formula form $s = . . .$ and $t = . . .$

c) Check your relationship for three more expansions. Add your results to the table.

d) Does your relationship work for decimals, e.g. $(a – 2.1)(a + 3.7)$? fractions, e.g., $\left(a - \frac{1}{2}\right)\left(a + \frac{3}{4}\right)$? Check and place in the table.

e) Rewrite your table in the form:

Expanded form **$a2 + sa + t$**	***s***	***t***	**Factorised form** **$(a + p)(a + q)$**	***p***	***q***
$a^2 + 7a + 12$	7	12	$(a + 4)(a + 3)$	4	3
			$(a + 1)(a + 2)$		
			$(a + 2)(a + 4)$		
			$(a - 2)(a + 5)$		
			$(a - 3)(a + 4)$		
			$(a - 2)(a - 5)$		

If you know the expanded form, for example $a^2 + 7a + 12$, how can you find the factorised form $(a + p)(a + q)$? In other words how can you find the value of p and the value of q?

Make a conjecture.

f) Check whether your conjecture works for $a^2 + 14a + 33$ by first completing the multiplication table.

×	a	. .
a	a^2	. .
. .	. .	33

Write down the factorisation of $a^2 + 14a + 33 = (a + ..)(a + ...)$

g) Do the same for:

(i) $a^2 + 7a + 10$ (ii) $a^2 + 11a + 10$ (iii) $a^2 - a - 6$ (iv) $a^2 + a - 12$

h) Does your method also work for the difference of two squares?
Try $b^2 - 25$.

×	b	. .
b	b^2	. .
. .	. .	25

i) Does your method always work?
Try to factorise $c^2 - 2c - 6$, or $c^2 - 4c + 3.75$ or $x^2 + 4$.
Discuss when your method works and when it does not work.

Take some more practice with these

1. Complete the multiplication tables and write down the factorisation of the expression.

a) $a^2 + 8a + 15$

$\times$	a	. .
a	a^2	. .
. .	. .	15

$a^2 + 8a + 15 = (a \text{ ....})(a \text{ ....})$

b) $b^2 - 12b + 35$

$\times$	b	. .
b	b^2	. .
. .	. .	35

$b^2 - 12b + 35 = (b \text{ ....})(b \text{ ....})$

c) $c^2 - 6c + 5$

$\times$	b	. .
b	b^2	. .
. .	. .	35

$c^2 - 6c + 5 = (c \text{ ....})(c \text{ ....})$

d) $d^2 - 6d - 7$

$\times$	d	. .
d	d^2	. .
. .	. .	$^-7$

$d^2 - 6d - 7 = (\qquad)(\qquad)$

e) $e^2 - e - 56$

$\times$	e	. .
e	e^2	. .
. .	. .	$^-56$

$e^2 - e - 56 = (\qquad)(\qquad)$

f) $f^2 + 21f - 72$

$\times$	f	. .
f	f^2	. .
. .	. .	$^-72$

$f^2 + 21f - 72 = (\qquad)(\qquad)$

2. Factorise

a) $x^2 + 11x + 24$ b) $y^2 + 14y + 40$ c) $x^2 + 14x + 45$
d) $z^2 - 4z - 21$ e) $a^2 - a - 30$ f) $b^2 - 13b + 30$
g) $c^2 - 13c - 30$ h) $d^2 + 13d - 30$

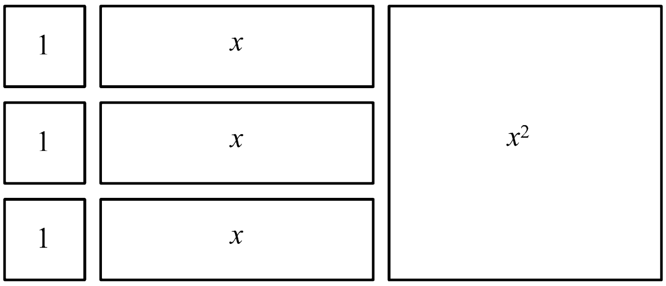
Self mark exercise 6

1. Work through the pupils' worksheet outlined above to factorise $x^2 + ax + b$.

Check your answers at the end of this unit.

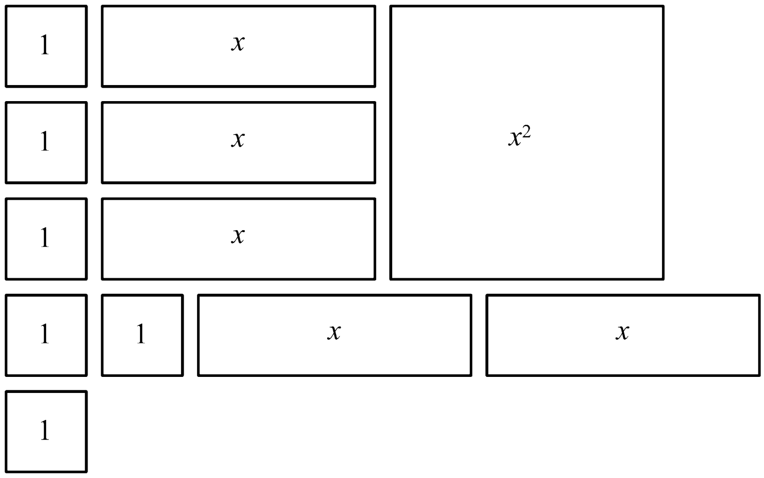
II. Using algebra tiles in factorisation of trinomials $ax^2 + bx + c$

Algebra tiles can be used as a concrete representation of factorisation. A good supply of tiles representing the unit, x and x^2 are needed.



To factorise for example $x^2 + 5x + 6$, one algebra tile representing x^2, 5 tiles representing x and 6 unit tiles are to be placed such that they form a rectangle (*see next page*). The dimensions of the rectangle will give the factors. Cut out tiles from pages 40 and 41.

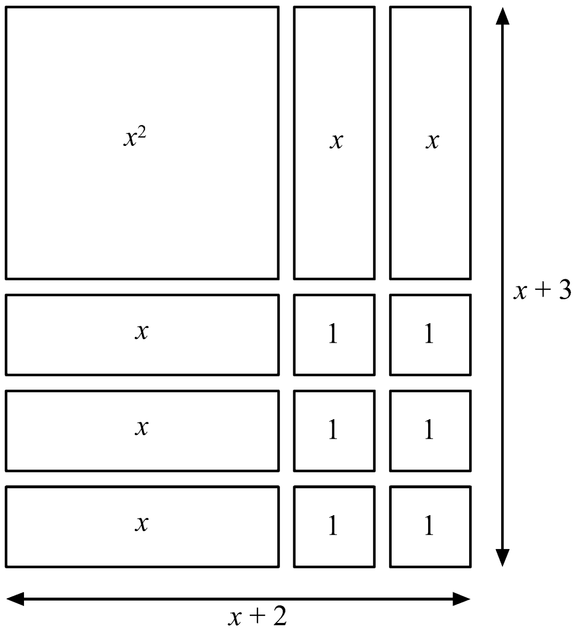
Given tiles



The given tiles are to be placed such that they form a rectangle. The length and width will be the required factors.

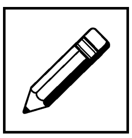
This is illustrated in the following diagram:

Placed to form a rectangle



The sides of the rectangle measure $(x + 3)$ and $(x + 2)$.

This illustrates the factorisation of $x^2 + 5x + 6 = (x + 2)(x + 3)$.



Self mark exercise 7

1. Use your algebra tiles to factorise the following trinomials
   a) $2x^2 + 3x + 1$
   b) $3x^2 + 4x + 1$
   c) $4x^2 + 8x + 3$

Check your answers at the end of this unit.

The restrictions of the above model are:

(i) positive integral coefficients are needed and

(ii) relatively small coefficient are needed ($10x^2 + 17x + 3$ would need 30 pieces!)

The first restriction can be overcome by using different coloured pieces to represent positive (white) and negative (shaded) terms.

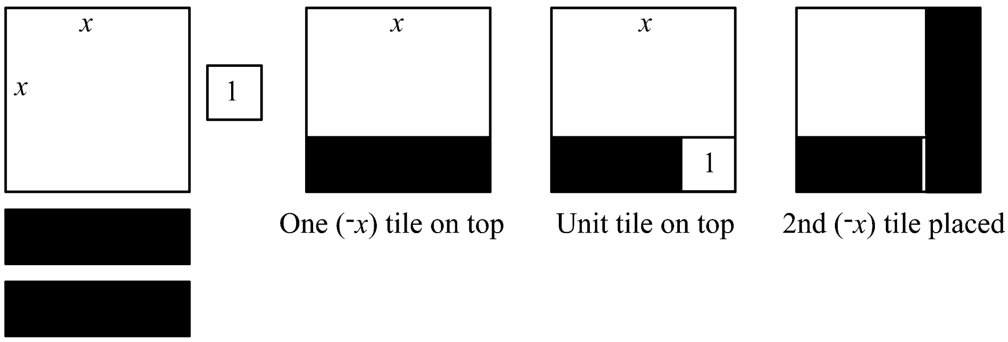
‘Negative’ regions are to be placed on top of a ‘positive’ region cancelling out the region. If need be, equal positive regions and negative regions can be added to the diagram.

Use your tiles to practice with the following cases:

(i) The diagram illustrates how to factorise $x^2 - 2x + 1$ and $x^2 + x - 2$.

You are to use one x^2 tile, two (^-x) tiles and one 1 tile to form a rectangle. This can be done as illustrated.

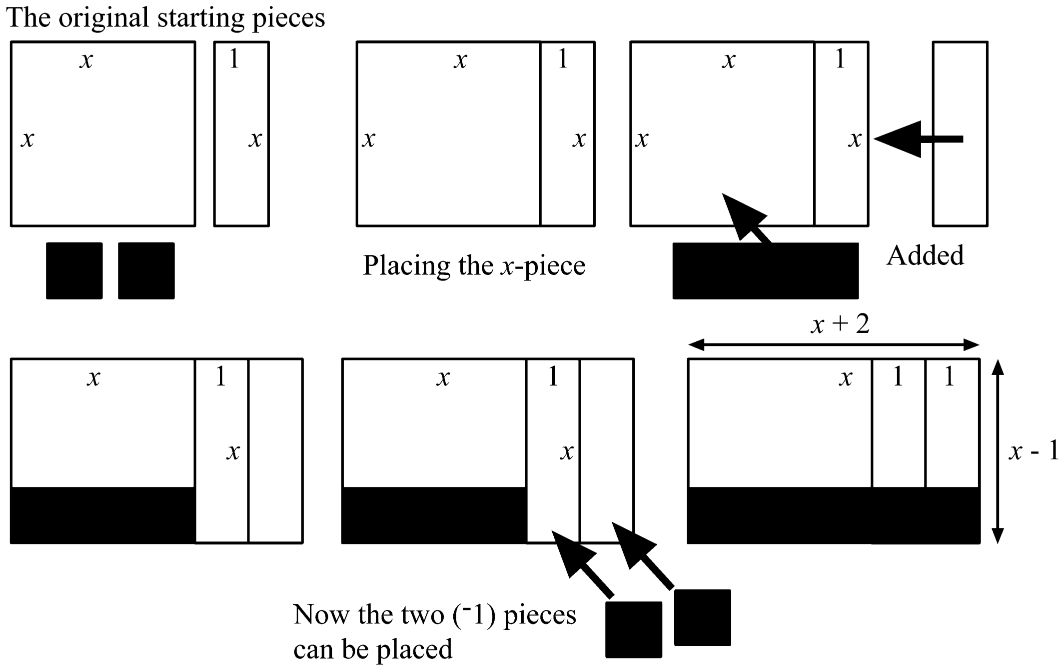
The original pieces



The unshaded area in the last diagram is $(x - 1)^2$

(ii) The next diagram illustrates how to factorise $x^2 + x - 2$.

You are to use one x^2 tile, one x tile and the two $(^-1)$ tiles to form a rectangle. This can be done as illustrated, with adding a (x) tile and a (^-x) tile.



$x^2 + x - 2 = (x + 2)(x - 1)$

Self mark exercise 8

Use your algebra tiles to factorise the following trinomials:

a) $x^2 + 2x - 3$

b) $x^2 - 2x - 3$

c) $2x^2 + 3x - 2$

Check your answers at the end of this unit.

The working with algebra tiles to factorise might be unfamiliar to you. When at school you might have been introduced to factorisation by a different method. Therefore your first impression might be that this approach is ‘difficult’. Remember that any new method introduced to pupils is ‘difficult’ to them and that the method they learn to work with becomes ‘easy’ as compared to another method they might be introduced to later. Research has indicated that pupils introduced to factorisation by using concrete manipulatives (tiles) prefer the concrete approach over a more abstract sum/product method.

The concrete working with tiles can be gradually replaced by a more abstract approach if pupils are ready for it. However the tiles should be available at all times so pupils can use them if they want. A more permanent set—wood, plastic, plasticized cardboard—is a valuable teaching/learning aid. The set can also be used on the overhead projector. The more abstract method was described in the previous section: the sum/product algorithm.

Unit 3, Practice activity

1. Try out in your class the activity for pupils to discover the factorisation of $x^2 + ax + b$ and the consolidation exercise.

   Write an evaluative report.

2. Try out the use of the algebra tiles in your class in the factorisation of trinomials.

   Write an evaluative report.

   Compare the use of the concrete manipulatives with the more abstract 'sum/product' method (use different classes!).

   Which method has your preference? Justify.

   Which method has pupils' preference?

3. Develop and try out a worksheet to assist pupils in the learning of the factorisation of $ax^2 + bx + c$. Justify the structure of your worksheet and the type of questions included.

   Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

Section C: Fallacies

Pupils do at times succeed without real understanding of the concepts: they have operational understanding i.e. they know how to apply certain algorithms without knowing why they work. Going through fallacies is a way to make pupils reflect on each and every step taken in an algebraic argument. It helps to sharpen their relational understanding.

Here are some examples of "algebraic proofs". The question is: What is wrong in the following 'proofs' as the outcome is clearly false?

(i) ALL people are the same age.

Proof: If my age is x years and your age is y years then our average age is $m = \frac{1}{2}(x+y)$.

so now $2m = x + y$

$$
\begin{aligned}
2m(x - y) &= (x + y)(x - y) \\
2mx - 2my &= x^2 - y^2 \\
y^2 - 2my &= x^2 - 2mx \\
y^2 - 2my + m^2 &= x^2 - 2mx + m^2 \\
(y - m)^2 &= (x - m)^2 \\
y - m &= x - m \\
y &= x
\end{aligned}
$$

So we are the same age.

(ii) Negative numbers are the same as positive numbers i.e. $^{-}a = {}^{+}a$ because:

$(^{-}a)^2 = (+a)^2$ Taking the square root of both sides

$^{-}a = +a$

(iii) 1 = 2

Proof:

Let $a = b$

Then $a^2 = ab$ (multiply both sides by a)

$a^2 - b^2 = ab - b^2$ (subtract b^2 from both sides)

$(a - b)(a + b) = b\,(a - b)$ (factorise)

$a + b = b$ (divide both sides by $a - b$)

But as $a = b$: $a + a = a$

$2a = a$ (divide by a)

$2 = 1$

(iv) $\frac{1}{4}$ m = 25 cm

Taking the square root at both sides gives

$\frac{1}{2}$ m = 5 cm

Self mark exercise 9

Clearly identify the error in each of the above ‘proofs’.

Check your answers at the end of this unit.

Unit 3, Practice activity

1. There are numerous algebraic fallacies. Most of them are based on the two erroneous ideas illustrated above (i) ignoring that the square root of a number or expression is to be a non negative value (ii) division by 0 is undefined, and (iii) a few based on operation on the numbers only and ignoring units.

   Find or construct at least three algebraic fallacies.

2. Present fallacies to your pupils. In groups they are to discuss what is wrong.

   Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

Summary

This unit has continued to build algebraic concepts through examples and working models (tiles), and has added the use of fallacies to your teaching arsenal. It achieved realism—meaning applicability to the present or future lives of most students—by *avoiding* any suggestion of using factorising in the workplace.

Unit 3: Answers to self mark exercises

Self mark exercise 1

1. Rectangles with dimensions 1 cm × 24 cm, 2 cm × 12 cm,
   $\frac{1}{2}$ cm × 48 cm, $\frac{1}{3}$ cm × 72 cm, $\sqrt{2}$ cm × 12$\sqrt{2}$ cm, etc. etc.
   In other words there are an infinite number of rectangles enclosing 24 cm^2.
2. To list but a few:

   $1 \times (6a + 9)$, $2 \times (3a + 4.5)$, $3 \times (2a + 3)$, $1.5 \times (4a + 6)$,
   $6(a + 1.5)$, $^-3(^-2a - 3)$, $100(0.06a + 0.09)$, $a \times (6 + \frac{9}{a})$, ...

Self mark exercise 2

1. a) $2 \times (3a + 4.5)$, $1.5 \times (4a + 6)$, $6(a + 1.5)$, $100(0.06a + 0.09)$, $a \times (6 + \frac{9}{a})$

   b) $1 \times (6a + 9)$

   c) NO there remain two possibilities, namely $3(2a + 3)$ and $^-3(^-2a - 3)$, both meeting the set conditions: factorised over the integers and hcf is outside the brackets.

2. a) 4 b) 3 c) $15x$ d) $2x^2$
3. a) $4x$ b) $3a^2$ c) $6x^2$ d) $4x^2$
4. a) $4xy$ b) $3ab^2$ c) $5xy^2$ d) $3x2y^3z$

Self mark exercise 3

N.B. Area diagrams have not been included here.

1. Question 1

   a) $4p + 12 = 4(p + 3)$ b) $5q + 10 = 5(q + 2)$ c) $6p + 9 = 3(2p + 3)$

   d) $8a + 16 = 8(a + 2)$ e) $c^2 + 3c = c(c + 3)$ f) $9pq - 6p = 3p(3q - 2)$

   Question 2

   a) $3(a + 2)$ b) $6(2a + b)$ c) $2(2x - 3y)$

   d) $5(x + 2y)$ e) $4(a^2 + 3b^2)$ f) $2a(3a - 1)$

   g) $4(2y - 3x)$ h) $2(12a - 19b)$ i) $2(x + 2y - 3)$

Question 3

a) $(a + 2)$ b) $6b(2 + b)$ c) $2x(2x - 3)$

d) $5x^2(x + 2)$ e) $4b(b + 3)$ f) $2a(3a - 1)$

g) $4y^2(2y - 3)$ h) $4(1 - abc^3)$ i) $x(x^2 + x - 1)$

Question 4

a) $3ab(b + 2)$ b) $6a^2b(2a + 1)$ c) $2y^2(2x - 3)$

d) $5x2y(xy + 2)$ e) $4b^2(a^2 + 3)$ f) $2ab(3a - b)$

g) $4xy^3(2 - 3x)$ h) $4ab(1 - c^3)$ i) $2xy(xy - 3x + 4y)$

2. Negative numbers as factors do not fit in the model.

Self mark exercise 4

Tables have not been included here.

1. a) $8a + 24 = 8(a + 3)$ b) $12b - 9 = 3(4b - 3)$ c) $t^2 - 8t = t(t - 8)$

2. a) $4(a + 3)$ 2b) $5(p - 6q)$ c) $^-3(d - 3e) = 3(^-d + 3e)$

d) $^-a(a + 4) = a(^-a - 4)$ e) $3a(^-b - 2b^2 + 3a) = {}^-3a(b + 2b^2 - 3a)$

Self mark exercise 5

1. h) $3 - 2 = (\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

2. The factorisations are correct. Pupil should be allowed to use any of the forms but should be made aware that the expressions are equivalent (if he/she is not aware of that). To set task to "show that the expressions are equivalent" would be a useful exercise in algebra.

3. a) $(7 + a)(7 - a)$ b) $(b + 0.5)(b - 0.5)$ c) $(3c + 4)(3c - 4)$

d) $(1 + 1.5d)(1 - 1.5d)$ e) $(98 + 2)(98 - 2)$ f) $(54 + 46)(54 - 46)$

g) $\left(\frac{2}{3} + \frac{1}{3}\right)\left(\frac{2}{3} - \frac{1}{3}\right)$ h) $(9.8 + 0.2)(9.8 - 0.2)$

i) $2(a^2 - 16) = 2(a + 4)(a - 4)$ j) $5(b + 3)(b - 3)$

4. a) $\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$ b $\left(\frac{2}{3x} + \frac{4y}{5}\right)\left(\frac{2}{3x} - \frac{4y}{5}\right)$

c) $(x^2 + 1)(x + 1)(x - 1)$

d) $(a + b + a - b)(a + b - a + b)$ $[= 2a \,.\, 2b = 4ab]$

e) $(x + 5 + 3)(x + 5 - 3) = (x + \ )(x + 2)$

f) $([3x - 1] + [4x + 5])\ ([3x - 1] - [4x + 4]) = (7x + 4)(^-x - 5)$

g) $(a^2 + b^2)(a + b)(a - b)$ h) $(x + \sqrt{3})(x - \sqrt{3})$ i) $(x\sqrt{2} + \sqrt{5})(x\sqrt{2} - \sqrt{5})$

Self mark exercise 6

1. a) See e

   b) $s = \text{P} + q,\ t = p \times q$

   e)

Expanded form $a^2 + sa + t$	s	t	Factorized form $(a + p)(a + q)$	p	q
$a^2 + 7a + 12$	7	12	$(a + 4)(a + 3)$	4	3
$a^2 + 3a + 2$	3	2	$(a + 1)(a + 2)$	1	2
$a^2 + 6a + 8$	6	8	$(a + 2)(a + 4)$	2	4
$a^2 + 3a - 10$	3	$^-10$	$(a - 2)(a + 5)$	$^-2$	5
$a^2 + a - 12$	1	$^-12$	$(a - 3)(a + 4)$	$^-3$	4
$a^2 - 7a + 10$	$^-7$	10	$(a - 2)(a - 5)$	$^-2$	$^-5$
$a^2 + 1.6a + 7.77$	1.6	7.77	$(a - 2.1)(a + 3.7)$	$^-2.1$	3.7
$a^2 + \frac{1}{4}a - \frac{3}{8}$	$\frac{1}{4}$	$-\frac{3}{8}$	$(a - \frac{1}{2})(a + \frac{3}{4})$?	$-\frac{1}{2}$	$\frac{3}{4}$

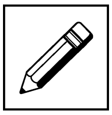
   f) $a^2 + 14a + 33 = (a + 11)(a + 3)$

   g) (i) $(a + 2)(a + 5)$
   (ii) $(a + 1)(a + 10)$
   (iii) $(a + 2)(a - 3)$
   (iv) $(a + 4)(a - 3)$

   i) The quadratics do not factor over the integers.

More practice

1. a) $(a + 3)(a + 5)$ b) $(b - 7)(b - 5)$ c) $(c - 1)(c - 5)$
   d) $(d - 7)(d + 1)$ e) $(e - 8)(e + 7)$ f) $(f - 3)(f -+ 24)$
2. a) $(x + 3)(x + 8)$ b) $(y + 4)(y + 10)$ c) $(x + 4)(x + 9)$
   d) $(z - 7)(z + 3)$ e) $(a - 6)(a + 5)$ f) $(b - 3)(b - 10)$
   g) $(c - 15)(c + 2)$ h) $(d + 15)(d - 2)$



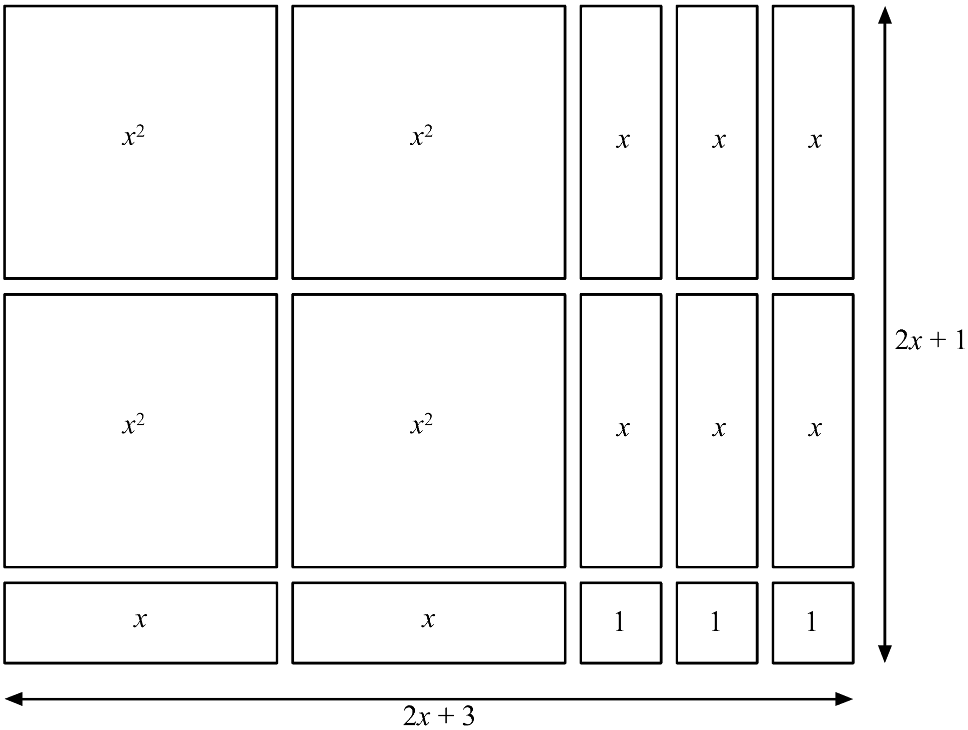
Self mark exercise 7

For example 1c:

You need to form a rectangle with 4 tiles of x^2, 8 tiles of x and 3 unit tiles. The diagram illustrates how the tiles are to be placed.

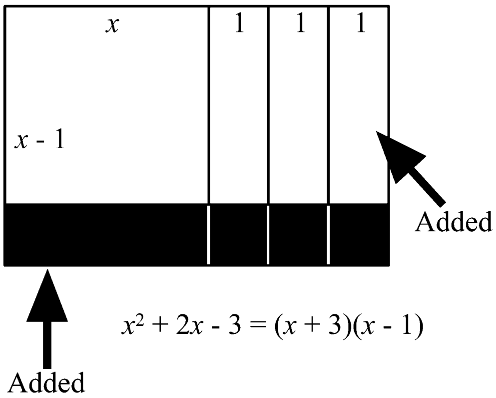
Hence $4x^2 + 8x + 3 = (2x + 3)(2x + 1)$.

Placed to form a rectangle



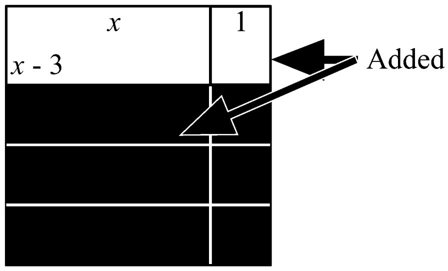
Self mark exercise 8

a)



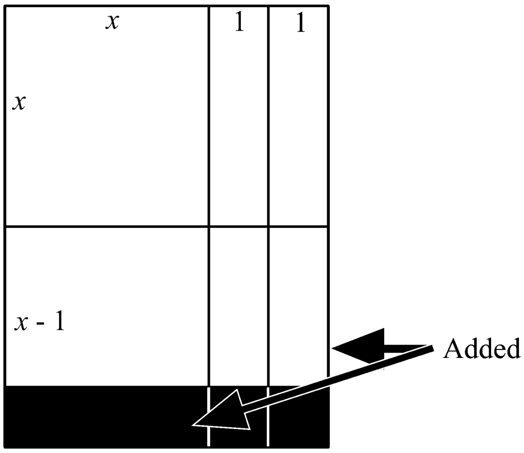
$x^2 + 2x - 3 = (x + 3)(x - 1)$

b)



$x^2 - 2x - 3 = (x - 3)(x + 1)$

c)



$2x^2 + 3x - 2 = (2x - 1)(x + 2)$

Self mark exercise 9

(i) and (ii) both move from a quadratic form to a linear by taking a square root. However by convention $\sqrt{p}$ is standing for the **non negative** number which when squared gives p.

In (I) up to the line $(y - m)^2 = (x - m)^2$ the working is correct.

If m is the average of x and y then either $x < m < y$ or $y < m < x$.

In the first case $x - m < 0$, in the second case $y - m < 0$

As the square root is to be a non-negative number

$(y - m)^2 = (x - m)^2$ is to be followed by either

$y - m = m - x$ (if $x < m < y$)

or by

$m - y = x - m$ (if $y < m < x$)

Both statements return to the (correct) starting line $x + y = 2m$.

In (ii) if $a > 0$: $\sqrt{(^-a)^2} = a$ (not ^-a) and $\sqrt{a^2} = a$ (not ^-a), as by convention the square roots are to be non negative.

If a < 0: $\sqrt{(^-a)^2} = {^-a}$ (not a) and $\sqrt{a^2} = {^-a}$ (not a), as by convention the square roots are to be non negative.

(iii) "Hiddenly" the equation is divided by 0 (in line three, dividing by $a - b$, but as $a = b$ you are dividing by 0, which is undefined).

(iv) is based on performing the operation (root extraction) only on the numbers and ignoring the units involved.

Unit 4: Binomial theorem and factor theorem

Introduction to Unit 4

Expanding and factorisation belong to the major manipulative techniques in algebra. The nature of algebra is to generalise. Expanding the product of $(ax + b)(cx + d)$ nearly automatically suggests to look at products of the form $(ax + b)(cx + d)(ex + f)$, $(Ax^2 + Bx + C)(ax^2 + bx + c)$, $(ax + by)^3$, etc. or in general the product of m polynomials. Similarly for factorisation: after having looked at trinomials one wonders "What about expressions with four terms, five terms, .. *n* terms?"

Purpose of Unit 4

In this unit you will look at extension work on expansion, the product of polynomials and binomial theorem, and extension work on factorisation, remainder and factor theorem. The intention is to extend and/or strengthen your content knowledge. The material covered in this unit is above the junior secondary level, although the high achiever could investigate some aspects, for example Pascal's triangle, covered in this unit. There is no attempt to relate these theorems to everyday life. Scientific calculators have supplanted all the practical uses of these theorems and their derived techniques.

Objectives

When you have completed this unit you should be able to:

- define a polynomial in one variable
- give examples and non-examples of polynomials in one variable
- state the degree of a polynomial
- multiply two polynomials by each other
- apply the binomial theorem for integer and fractional indices
- use Pascal's triangle in expansion of $(ax + by)^n$, where n is a positive integer less than 10
- define factorial n
- state the meaning of $\binom{n}{r}$
- write the binomial expansion for positive integer values of the exponent using factorial notation and $\binom{n}{r}$ notation.
- recognise Pascal's triangle pattern in various situations
- divide a polynomial of degree n by a polynomial of degree m where $(n > m)$
- use the remainder theorem to find the remainder of a polynomial when divided by a linear factor $ax + b$
- use the factor theorem to find a factor of a polynomial and factorise the polynomial completely

Time

To study this unit will take you about 10 hours.

Unit 4: Binomial theorem and factor theorem

Section A1: Product of polynomials

An expression of the form ax^n is called a **monomial** in one variable x, where a is a real number and n is a non negative integer.

ax^ny^m is a monomial in two variables x and y, where a is a real number and n and m are non negative integers.

The sum of monomials is called a **polynomial**. For one variable the general form is

$a_0 + a_1x + a_2x^2 + ... + a_nx^n$, coefficients are real numbers.

If $a_n \neq 0$, the one variable polynomial in x is said to be of **degree *n*** i.e. the highest index of x occurring with a non zero coefficient.

Two polynomials $A(x)$ and $B(x)$ can be multiplied by applying the 'extended' distributive law. The lay-out can take different formats.

Example:

(i) $(x^2 - 3x + 1)(2x^3 + x - 3)$

$= x^2(2x^3 + x - 3) - 3x(2x^3 + x - 3) + 1(2x^3 + x - 3)$

$= 2x^5 + x^3 - 3x^2 - 6x^4 - 3x^2 + 9x + 2x^3 + x - 3$

$= 2x^5 - 6x^4 + 3x^3 - 6x^2 + 10x - 3.$

(ii) The working can also be represented in a multiplication table:

$\times$	$2x^3$	x	$^-3$
x^2	$2x^5$	x^3	$^-3x^2$
^-3x	$^-6x^4$	$^-3x^2$	$9x$
1	$2x^3$	x	$^-3$

Taking like terms together gives: $2x^5 - 6x^4 + 3x^3 - 6x^2 + 10x - 3.$

As in the previous working, like terms have to be 'found'—they are not neatly standing below each other to allow convenient taking together of the like terms.

(iii) Lay out as with 'long multiplication' of numbers. Leaving 'space' for 'missing' powers, the multiplication allows one immediately to order the powers such that like terms are placed in the same column, making addition easier.

$$
\begin{array}{rrrrrr}
 & & 2x^3 & & +x & -3 \\
 & & & x^2 & -3x & +1 \\
\hline
 & & 2x^3 & & +x & -3 \\
 & {}^-6x^4 & & {}^-3x^2 & +9x & \\
2x^5 & & +x^3 & -3x^2 & & \\
\hline
2x^5 & -6x^4 & +3x^3 & -6x^2 & +10x & -3
\end{array}
$$

Self mark exercise 1

1. State the degree and the coefficient of x^3 for each of these polynomials:

   a) $2x^3 - 2x + 17$ b) $8 - 2x^2 + 5x^3$ c) $4x^4 - 2x^2 - 9x$

   d) 3 e) $3x^8 - 4x^6 - x^3 + 1$

2. Expand and write the answer in descending powers of x.

   a) $(2x^2 - 3x + 3)(3x^2 + x - 2)$ b) $(3 - x)(2 - x^2 + 4x^3)$

   c) $(2x^6 - 3x^2 - 2)(3x^6 + 4x^2 - 1)$ d) $(2x - 3)(4x + 5)(x^2 - 2x + 3)$

3. *a*) A polynomial P_n of degree n is multiplied by a polynomial P_m of degree m. What is the degree of the product polynomial?

b. A polynomial P_n of degree n is squared. What is the degree of the resulting polynomial?

c. A polynomial P_n of degree n is raised to the power t. What is the degree of the resulting polynomial?

Check your answers at the end of this unit.

Section A2: Binomial expansion

Check the expansion of powers of $(x + 1)$ listed below by multiplying the previous result by $(x + 1)$ to obtain the next line.

$(x + 1)^1 = x + 1$

$(x + 1)^2 = x^2 + 2x + 1$

$(x + 1)^3 = (x + 1)(x + 1)^2 = x^3 + 3x^2 + 3x + 1$

$(x + 1)^4 = (x + 1)(x + 1)^3 = x^4 + 4x^3 + 6x^2 + 4x + 1$

$(x + 1)^5 = (x + 1)(x + 1)^4 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

The triangle formed with the coefficients of the expansion of $(x + 1)^n$ is known as Pascal's triangle as it was intensively studied by the French mathematician Blaise Pascal (1623 - 1662). The triangle had already been described much earlier by Chinese mathematicians (1303 Chu Shih-chieh) and by Omar Khayyam (ca 1100, Persian poet and mathematician). The first rows, following from the above results, are as follows

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Self mark exercise 2

1. Expand

   a) $(x + 1)^6$

   b) $(x + 1)^7$

   c) Using your results of a and b write the next two rows in Pascal's triangle.

2. a) Study the triangle and see whether you can find a pattern to construct the next row from the previous one.

   b) Using the pattern write down two more rows in Pascal's triangle.

Check your answers at the end of this unit.

To find an expression for the expansion of $(x + 1)^n$ study the following pattern:

$(x + 1)^1 = x + 1$

$(x + 1)^2 = x^2 + 2x + 1$

$(x + 1)^3 = x^3 + 3x^2 + 3x + 1 = x^3 + \frac{3}{1}x^2 + \frac{3 \times 2}{1 \times 2}x + 1$

$(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1 = x^4 + \frac{4}{1}x^3 + \frac{4 \times 3}{1 \times 2}x^2 + \frac{4 \times 3 \times 2}{1 \times 2 \times 3}x + 1$

$(x + 1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 =$

$$x^5 + \frac{5}{1}x^4 + \frac{5 \times 4}{1 \times 2}x^3 + \frac{5 \times 4 \times 3}{1 \times 2 \times 3}x^2 + \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4}x + 1$$

$(x + 1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 =$

$$x^6 + \frac{6}{1}x^5 + \frac{6 \times 5}{1 \times 2}x^4 + \frac{6 \times 5 \times 4}{1 \times 2 \times 3}x^3 +$$

$$\frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4}x^2 + \frac{6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5}x + 1$$

$(x + 1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

$(x + 1)^8 = x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1$

Self mark exercise 3

1. In a similar way write down the coefficients in the expansion of $(x + 1)^7$ and $(x + 1)^8$.
2. Write down the expansion of $(x + 1)^n$ by generalising the pattern in (1).

Check your answers at the end of this unit.

The result inductively obtained in the self mark exercise 3, question 2 is known as the binomial expansion for positive integer index or binomial theorem.

$$(1 + x)^n = 1 + nx + \frac{n \times (n-1)}{1 \times 2}x^2 + \frac{n \times (n-1) \times (n-2)}{1 \times 2 \times 3}x^3 +$$

$$\frac{n \times (n-1) \times (n-2) \times (n-3)}{1 \times 2 \times 3 \times 4}x^4 + \ldots +$$

$$\ldots + \frac{n \times (n-1) \times (n-2) \times (n-3) \times (n-4) \times \ldots \times (n - \{r-1\})}{1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times r}x^r + \ldots + \ldots\ x^n$$

The coefficients in the expansion can be obtained from Pascal's triangle for lower values of n. For larger values of n the above formula is more economical to use.

Note the following:

(i) The number of terms in the expansion is $(n + 1)$.

(ii) The expansion can be written down in ascending or descending powers of x due to the symmetry in the coefficients.

(iii) the general term in the expansion x^r has as coefficient

$$\frac{n \times (n-1) \times (n-2) \times (n-3) \times (n-4) \times .... \times (n-\{r-1\})}{1 \times 2 \times 3 \times 4 \times 5 \times .... \times r}$$

(iv) No proof of the binomial theorem for positive integer index has been given. To prove the theorem for positive integer index proof by induction is generally used. This is not discussed in this unit.

(v) The expansion is symmetric as you can see in the structure of Pascal's triangle: you can write down the expansion in ascending or in descending powers of x using the same row in the triangle. The coefficients of x^r and x^{n-r} are the same.

(vi) The product $1 \times 2 \times 3 \times 4 \times 5 \times ..... \times n$ is written in mathematics as $n!$ and called 'n factorial'. There should be a factorial key on your calculator. For example: $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

$0!$ is taken as 1.

This allows a shorter notation for the binomial coefficient of the general term:

$$\frac{n \times (n-1) \times (n-2) \times (n-3) \times (n-4) \times ... \times (n-\{r-1\})}{1 \times 2 \times 3 \times 4 \times 5 \times ... \times r} =$$

$$\frac{n \times (n-1) \times (n-2) \times (n-3) \times (n-4) \times ... \times (n-\{r-1\}) \times (n-r) \times (n-r-1) \times ... \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times ... \times r \times (n-r) \times (n-r-1) \times ... \times \ 2 \times 1} =$$

$$\frac{n!}{r! \times (n-r)!} = \binom{n}{r}$$

Using the factorial notation the binomial theorem can be written as

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + ... + \binom{n}{r} x^r + ... + x^n$$

(vii) The expansion of $(1+x)^n$ can be used to obtain the expansion of the more general expression

$(p+q)^n$ as follows:

$$[p(1+\frac{q}{p})]^n = p^n(1+\frac{q}{p})^n = p^n\left\{1 + n\times\frac{q}{p} + \frac{n\times(n-1)}{1\times2}\left(\frac{q}{p}\right)^2 + \frac{n\times(n-1)\times(n-2)}{1\times2\times3}\left(\frac{q}{p}\right)^3 + \frac{n\times(n-1)\times(n-2)\times(n-3)}{1\times2\times3\times4}\left(\frac{q}{p}\right)^4 + \ldots + \frac{n\times(n-1)\times(n-2)\times(n-3)\times(n-4)\times\ldots\times(n-\{r-1\})}{1\times2\times3\times4\times5\times\ldots\times r}\left(\frac{q}{p}\right)^r + \ldots + \ldots\left(\frac{q}{p}\right)^n\right\} =$$

$$p^n + n\times p^{n-1}q + \frac{n\times(n-1)}{1\times2}p^{n-2}q^2 + \frac{n\times(n-1)\times(n-2)}{1\times2\times3}p^{n-3}q^3 + \frac{n\times(n-1)\times(n-2)\times(n-3)}{1\times2\times3\times4}p^{n-4}q^4 + \ldots + \frac{n\times(n-1)\times(n-2)\times(n-3)\times(n-4)\times\ldots\times(n-\{r-1\})}{1\times2\times3\times4\times5\times\ldots\times r}p^{n-r}q^r + \ldots + \ldots q^n$$

In factorial notation this becomes:

$(p+q)^n =$

$$p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \binom{n}{3}p^{n-3}q^3 + \ldots + \binom{n}{r}p^r q^{n-r} + \ldots + q^n$$

Note that in this expansion:

The number of terms in the expansion is $(n+1)$.

The general term is $p^{n-r}q^r$

The sum of the indices of p and q in each term is always n.

Due to the symmetry you can interchange p and q in the expression and by doing so the expansion is either in ascending powers of p (and descending powers of q) or in descending powers of p (and ascending powers of q).

Example: $(3x-4y)^3$ expanded in ascending powers of x (and hence descending powers of y) gives:

$$(^-4y)^3 + 3\times(3x)\times(^-4y)^2 + 3\times(3x)^2\times(^-4y) + (3x)^3 =$$

$$^-64y^3 + 144xy^2 - 108x2y + 27x^3$$

or in descending powers of x (and ascending powers of y):

$$(3x)^3 + 3\times(3x)^2\times(^-4y) + 3\times(3x)\times(^-4y)^2 + (^-4y)^3 =$$

$$27x^3 - 108x2y + 144xy^2 - 64y^3$$

Self mark exercise 4

Using the binomial expansion expand in descending powers of x.

1. $(1 + 2x)^4$ 2. $(1 - 2x)^3$ 3. $(1 + \frac{1}{2}x)^5$

4. $(2x - 3y)^4$ 5. $(x^2 + y^2)^3$ 6. $(\frac{1}{2}x + \frac{1}{3}y)^3$

7. Use the binomial expansion to obtain the value of $(71.02)^3$ to 2 decimal places accurately given that $71^3 = 357\,911$. Hint: expand $(71 + 0.03)^3$.

8. Expand $(1 + x - x^2)^5$ in ascending powers of x up to and including the term in x^3 by writing the expression as $\{1 +(x - x^2)\}^5$.

9. a) Find the coefficient of x^4 in the expansion of $(3x - 2)^{11}$.

   b) Find the coefficient of x^3 in the expansion of $(2 - 5x)^9$.

10. Find the term independent of x in the expansion of $(3x + \frac{2}{x})^{12}$.

11. Use the expansion of $(2 + x)^7$ to obtain correct to 3 decimal places the value of:

    a) $(2.08)^7$

    b) $(1.99)^7$

12. Find the value of the following binomial coefficients:

    a) $\binom{8}{3}$ b) $\binom{8}{5}$ c) $\binom{12}{4}$ d) $\binom{12}{8}$ e) $\binom{25}{13}$ f) $\binom{25}{12}$

    g) What does this suggest for $\binom{n}{r}$ and $\binom{n}{n-r}$?

    h) Prove your conjecture made in g.

Check your answers at the end of this unit.

The binomial expansion for $(1 + x)^n$ holds in general for all real values of n. For the proof you need Maclaurin's theorem which is covered in calculus courses. Here we will assume that the expansion can be applied and investigate whether it makes sense.

First we look at positive rational index. Remember that $x^{p/q} = \sqrt[q]{x^p}$ where p and q are positive integers.

Check that for $(1+x)^{1/3} = \sqrt[3]{1+x}$ you will get the following expansion (Note that the factorial notation has no meaning in this case and the original expression for the expansion is to be used).

$$1+\frac{1}{3}x+\frac{\frac{1}{3}\times(\frac{1}{3}-1)}{1\times 2}x^2+\frac{\frac{1}{3}\times(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\times 2\times 3}x^3+\ldots=$$

$$1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\ldots$$

Work out the next term in the expansion to confirm that the next term is

$$-\frac{10}{243}x^4$$

The series, unlike in the case for positive integer index, will never end as the coefficient of the powers of x will never become 0.

Does the infine series give reasonable results?

Let's check this for some values of x.

Checking for $x = 7$

$x = 7$ gives $(1+7)^{\frac{1}{3}} = \sqrt[3]{1+7} = \sqrt[3]{8} = 2$

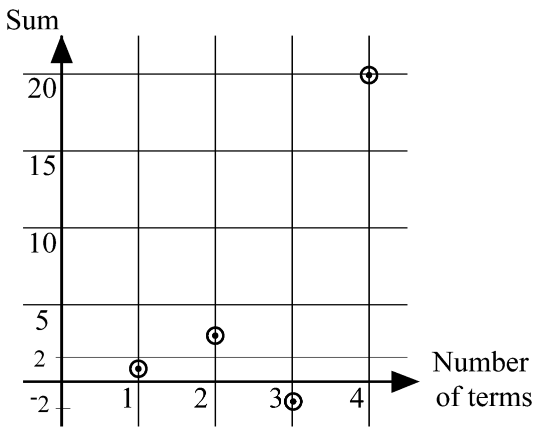
The expansion, for $x = 7$, gives

$$1+\frac{1}{3}\times 7-\frac{1}{9}\times 7^2+\frac{5}{81}\times 7^3-\frac{10}{243}\times 7^4\ldots=1+\frac{7}{3}-5\frac{4}{9}+21\frac{14}{81}-98\frac{196}{243}+\ldots$$

Tabulating the sum obtained against the number of terms used in the expansion we get:

Number of terms	Sum
1	1
2	3.33
3	-1.11
4	20.06
5	-78.74

This in no way comes closer to the expected value of 2. The more terms used in the expansion the further away from the value 2 one gets. The series is **divergent** for $x = 7$. This is illustrated in the graph below:



Checking $x = 2$

Expected outcome of the series is $(1+2)^{\frac{1}{3}} = \sqrt[3]{1+2} = \sqrt[3]{3} \approx 1.442$

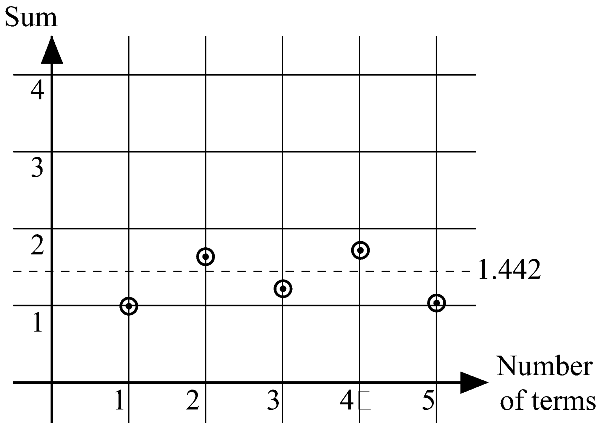
The expansion, for $x = 2$, gives

$$1+\frac{1}{3}\times 2-\frac{1}{9}\times 2^2+\frac{5}{81}\times 2^3-\frac{10}{243}\times 2^4\ldots\ldots = 1+\frac{2}{3}-\frac{4}{9}+\frac{40}{81}-\frac{160}{243}+\ldots\ldots$$

Tabulating the sum obtained against the number of terms used in the expansion we get:

Number of terms	Sum
1	1
2	1.67
3	1.22
4	1.72
5	1.06

As in the previous case the values of the sum do not come closer to the expected value of 1.442. To get closer to the answer the terms in the expansion should become smaller and smaller (in absolute value). The graph below illustrates that the values do not get closer to 1.442. The series is **divergent** for $x = 2$



Checking $x = 1$

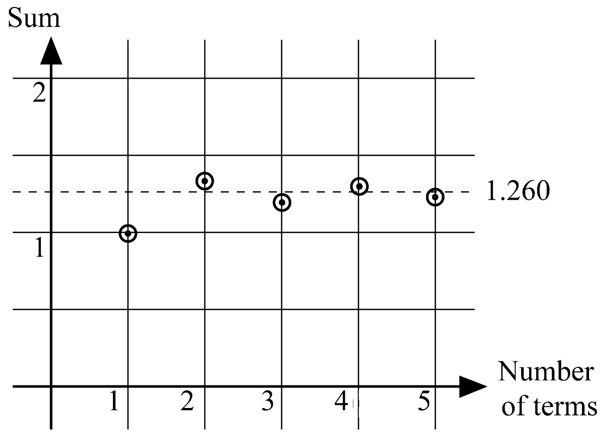
Expected outcome of the series is $(1+1)^{\frac{1}{3}} = \sqrt[3]{1+1} = \sqrt[3]{2} \approx 1.260$

The expansion, for $x = 1$, gives $1+\frac{1}{3}-\frac{1}{9}+\frac{5}{81}-\frac{10}{243}\ldots\ldots$

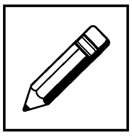
Tabulating the sum obtained against the number of terms used in the expansion we get:

Number of terms	Sum
1	1
2	1.33
3	1.22
4	1.28
5	1.24

Presented in a graph



The values are alternatively a little above or below the true sum, but getting closer each time. The series **converges** to the value $\sqrt[3]{2} \approx 1.260$.

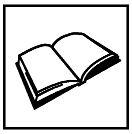


Self mark exercise 5

Use the expansion of $(1+x)^{\frac{1}{3}} = \sqrt[3]{1+x} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + ...$ to check whether the series converges or diverges if

1. $x = \frac{-7}{8}$ 2. $x = 0.1$ 3. $x = {}^-1$

Check your answers at the end of this unit.



The result of investigating the expansion of $(1+x)^{\frac{1}{3}} = \sqrt[3]{1+x}$ is that the series diverges for values of $x > 1$ or $x < {}^-1$ and is convergent for ${}^-1 < x < 1$. This is generally true for the expansion of

$(1+x)^n$, where n is a positive rational number. The series is convergent for ${}^-1 < x < 1$ and divergent for values of x outside this range.

The binomial expansion has lost its utility, even for scientists and engineers, as calculators are easily available now. In the days when no calculators were available, the binomial expansion allowed the computation of surds to any degree of accuracy required. Tables of square roots, cube roots etc. were constructed using the binomial expansion.

For example, using the expansion

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + ...$$

and taking $x = \frac{2}{9}$ the left hand side gives us $\sqrt{1+\frac{2}{9}} = \sqrt{\frac{11}{9}} = \frac{1}{3}\sqrt{11}$

The right hand side (the first terms of the series) gives

$$1+\frac{1}{2}\times\frac{2}{9}-\frac{1}{8}\times\left(\frac{2}{9}\right)^2+\frac{1}{16}\times\left(\frac{2}{9}\right)^3-\frac{5}{128}\times\left(\frac{2}{9}\right)^4+\ldots=$$

$$1+0.11111-0.00617+0.00068-0.000095+\ldots.$$

Hence $\frac{1}{3}\sqrt{11}=1.10553$ to 5 decimal places and $\sqrt{11}=3.3166$ to 4 decimal places.

Remember that in order to obtain a final result to 4 decimal places accurately, in the working you have to use (at least) one more decimal place and round to 4 decimal places in the last step.

The expression $(p+q)^n$ cannot be expanded using the expansion you obtained when n is a positive integer, if n is a (positive) rational number. More general: the expansion for non positive integer values of n applies to the form $(1+t)^n$ only i.e. the expression in the brackets is to start with 1. For example, to expand the expression $(4+x)^{\frac{1}{2}}$ it is to be restructured into the format

$(1+t)^n$ by writing $(4+x)^{\frac{1}{2}}=[4(1+\frac{x}{4})]^{\frac{1}{2}}=2(1+\frac{x}{4})^{\frac{1}{2}}$ before applying the binomial theorem.

Self mark exercise 6

1. Expand $\sqrt[5]{1+x}$ in ascending power of x up to and including the term in x^3. Use the expansion and take $x = 0.02$ to find to 4 decimal places accurately $\sqrt[5]{1.02}$
2. Expand $\sqrt{5x+1}$ in ascending power of x up to and including the term in x^3. Use the expansion and take $x=\frac{1}{100}$ to find to 4 decimal places accurately $\sqrt{105}$
3. Expand $\sqrt[3]{1-4x}$ in ascending power of x up to and including the term in x^3. Use the expansion and take $x=\frac{1}{100}$ to find to 4 decimal places accurately $\sqrt[3]{15}$
4. Expand $\sqrt{1-3x}$ in ascending power of x up to and including the term in x^3. Use the expansion and take $x=\frac{1}{30}$ to find to 4 decimal places accurately $\sqrt{10}$

Self mark exercise 6 continued on following page

Self mark exercise 6 continued

5. Expand $(1 - 2x)^{\frac{3}{2}}$ in ascending power of x up to and including the term in x^3. Use the expansion and take $x = \frac{1}{8}$ to find to 4 decimal places accurately $\sqrt{3}$
6. Expand $\sqrt{2 + x}$ in ascending power of x up to and including the term in x^3.
7. Assume that the expansion of $(1 + x)^n$ is valid for negative integers and rational numbers

   Find the first 5 terms in the expansion in ascending powers of x of the following expressions and investigate the range of values of x for which the expansion is valid.

   a) $(1 + x)^{-1}$

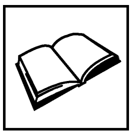
   b) $(1 + x)^{-2}$

   c) $(1 + x)^{-\frac{1}{2}}$
8. Expand $(1 + x)^{-3}$ in ascending powers of x up to and including the term in x^4. Use the expansion and take $x = -0.01$to find to 4 decimal places accurately $(0.99)^{-3}$.
9. Expand in ascending powers of x up to and including the term in x^4:

   a) $(1 + 3x)^{-2}$ b) $(1 - 4x)^{-5}$ c) $(2 - x)^{-2}$
10. Expand in ascending powers of x up to and including the term in x^3:

    a) $(1 - x)^{-\frac{1}{2}}$ b) $(1 + 2x)^{-\frac{1}{2}}$

Check your answers at the end of this unit.



In summary: If the binomial expansion of $(1 + x)^n$ leads to an infinite series, the series

(i) if $n > 0$ converges for $-1 < x < 1$

(ii) if $-1 < n < 0$ converges for $-1 < x < 1$

(iii) if $n < -1$ converges for $-1 < x < 1$

Section B1: Investigating Pascal's Triangle

Pascal's triangle has been studied for many years and mathematicians continue to find interesting patterns and situations related to or leading to Pascal's triangle.

Reading along diagonals you might have spotted the counting numbers 1, 2, 3, 4, .. and the triangular numbers 1, 3, 6, 10, 15, ... The following investigations all relate to Pascal's triangle.

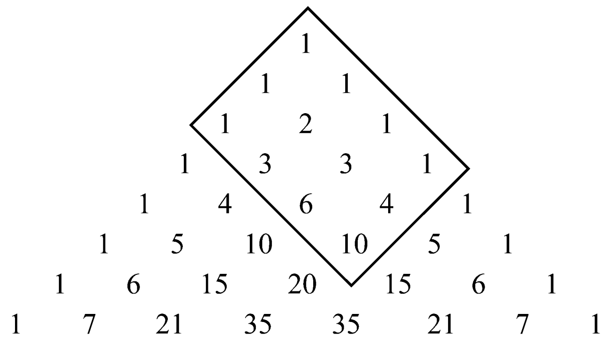
Problem Solving 1

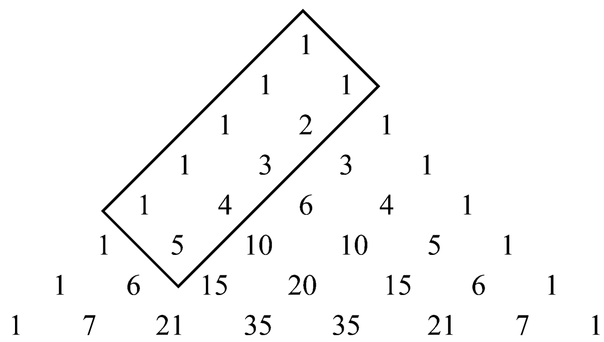
Investigate the following:

1. The sum of the numbers in the nth row of the Pascal's triangle.

Number of row	Sum of numbers
1	1
2	2
3	4
4	8

2. Find the sum of all the numbers in Pascal's triangle contained in rectangles with as one corner the top number 1. Relate it to one of the numbers outside the rectangle. For example:





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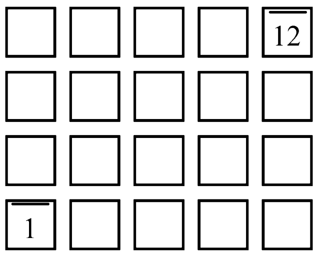
3. Starting at the top how many downwards paths can you take to spell "MATHEMATICS"?

```
            M
          A   A
        T   T   T
      H   H   H   H
    E   E   E   E   E
  M   M   M   M   M   M
    A   A   A   A   A
      T   T   T   T
        I   I   I
          C   C
            S
```

4. Find the sum of the numbers on the perimeter of Pascal triangles with 2, 3, 4, ... *n* rows.

1	1	1
1 1	1 1	1 1
	1 2 1	1 2 1
		1 3 3 1
2 rows	3 rows	4 rows
Sum 3	Sum 7	Sum 13

5. Lebogang lives in house 1 in a quarter where the houses are placed as shown in the diagram. There are paths between the houses. Her friend lives in house number 12. In how many different ways can Lebogang reach her friend's house by always moving either right or up in the diagram? She leaves by the front door of her own house and enters her friend's house by the front door (bold line in diagram).



Check your answers at the end of this unit.

Section B2: Pascal's Pyramid

Pascal's triangle can be used to determine the coefficients of a binomial expansion $(a + b)^n$. To obtain the coefficients in the expansion of trinomials $(a + b + c)^n$ Pascal's triangle can be extended to Pascal's pyramid. Generate first some data by using multiplication of polynomials to find the expansion of $(a + b + c)^2$, $(a + b + c)^3$ and $(a + b + c)^4$.

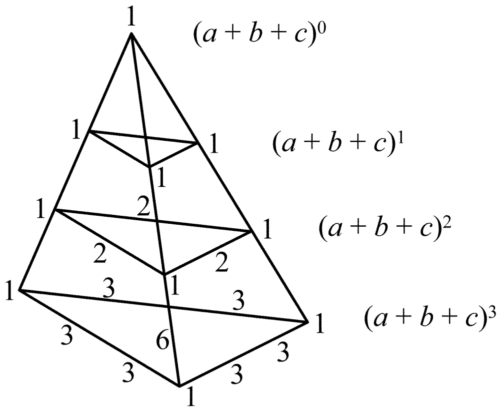
Self mark exercise 7

1. Obtain the expansion of

   a) $(a + b + c)^2$ b) $(a + b + c)^3$ c) $(a + b + c)^4$.

Check your answers at the end of this unit.

The coefficients in the expansion of $(a + b + c)^n$ can be represented by sections of a triangular pyramid.



The first expansion $(a + b + c)^0$ has the single coefficient 1, represented by the vertex of the pyramid.

The next expansion $(a + b + c)^1$ with coefficients 1, 1, 1 is represented by the next section of the pyramid with the 1 on the lateral edges of the pyramid.

The next layer is to represent the coefficients in the expansion of $(a + b + c)^2$.

The numbers on the outer edges of each layer (the numbers between the vertices) are the sum of the two numbers directly above it in the previous layer. The number(s) in the interior of the triangle is the sum of the three numbers above it in the previous layer.

First layer

```
   1
1     1
```

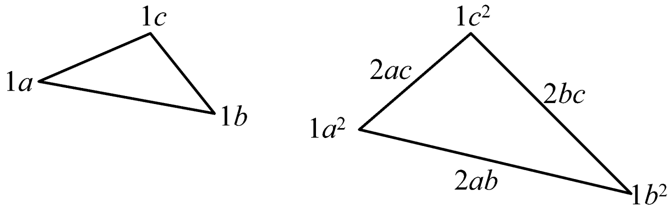
Second layer

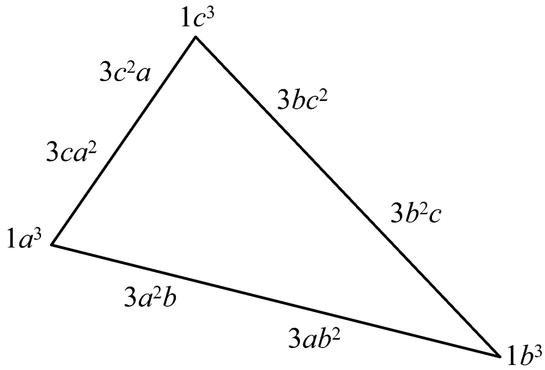
```
      1
   2     2
1     2     1
```

Third layer

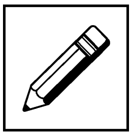
```
      1
   3     3
3     6     3
```

To assign these coefficients to the correct term in the expansion consider the pattern in the following diagrams:





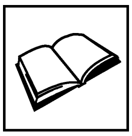
Looking, for example, at the third layer: starting from the vertex with a^3 moving towards the vertex with b^3, the exponents of *a* decrease by 1 in each step and the exponents of *b* increase by 1. This applies similar along the other sides of the triangle. The coefficient in the middle, 6, was the result of adding 2 + 2 + 2 from the previous layer. These were the coefficients of *ab*, *bc*, and *ca*. These have as lowest common multiple *abc*—the term in the middle of the next layer.



Self mark exercise 8

1. Using the third layer section represent the coefficients of the fourth layer in a triangle and assign the coefficient to the appropriate terms.
2. What do you notice about the numbers along the sides of the triangles in each layer?

Check your answers at the end of this unit.



The observation in question 2 of the Self mark exercise 8 allows for an alternative way of obtaining the coefficients.

The Pascal triangle ending with the coefficients 1 4 6 4 1 looks as follows:

```
            1
         1     1
      1     2     1
   1     3     3     1
1     4     6     4     1
```

The lower side is as required in our 4th pyramid layer.

Multiplying the first row by 1, the second by 4, the third by 6, the fourth by 4 and the fifth by 1 to obtain the numbers as required on the sides of the pyramid triangle generates the required layer of the pyramid.

1	x1 →	1
1 1	x4 →	4 4
1 2 1	x6 →	6 12 6
1 3 3 1	x4 →	4 12 12 4
1 4 6 4 1	x1 →	1 4 6 4 1

Self mark exercise 9

1. Starting with the appropriate Pascal's triangle, obtain the coefficients for the expansion of $(a + b + c)^5$
2. Obtain the expansion of

   a) $(a + b + c)^5$ b) $(a + 3b + 2c)^3$ c) $(a + 2b + c)^4$.
3. Investigate how the coefficients in the expansion of $(a + b + c)^n$ can be expressed using factorial notation.
4. Investigate the 'multinomial theorem' i.e. the expansion of $(a_1 + a_2 + a_3 + ... + a_r)^n$

Check your answers at the end of this unit.

Section C1: Dividing polynomials

Polynomials can be divided using long division. For example to divide $6x^3 – 13x^2 + 4x + 5$ by $(2x + 1)$ you could work as follows:

$$
\begin{array}{r l}
 & 3x^2 - 8x + 6 \\
2x+1 \,) & 6x^3 - 13x^2 + 4x + 5 \\
 & \underline{6x \quad + 3x^2 \quad + 4x} \\
 & {}^-16x^2 \quad + 4x \\
 & \underline{{}^-16x^2 \quad - 8x} \\
 & 12x - 5 \\
 & \underline{12x + 6} \\
 & {}^-1
\end{array}
$$

Consider only the $2x$ in the divisor and the first term of the polynomial $6x^3$.

$2x$ into $6x^3$ goes $3x^2$ times.

Multiply $(2x + 1)$ by $3x^2$ to give $6x^3 + 3x^2$.

Subtract this from $6x^3 – 13x^2$, which will give you ${}^-16x^2$.

$2x$ into ${}^-16x^2$ goes ${}^-8x$ times.

Multiply $(2x + 1)$ by $^{-}8x$ to give $^{-}16x^2 – 8x$.

Bring the $+ 4x$ down and subtract $^{-}16x^2 – 8x$ from $^{-}16x^2 + 4x$ to give you $+ 12x$.

$2x$ into $+ 12x$ goes 6 times. Multiplying $(2x + 1)$ by 6 leads to $12x + 6$.

Bring down +5 and subtracting leads to the remainder $^{-}1$.

Hence $(6x^3 – 13x^2 + 4x + 5) \div (2x + 1)$ gives as quotient $(3x^2 – 8x + 6)$ and as remainder $^{-}1$.

Compare with $13 \div 3$ which gives 4 remainder 1.

This can be written as $13 \div 3 = 4 + \frac{1}{3}$

or $13 = 3 \times 4 + 1$, i.e., (dividend) = (divisor) $\times$ (quotient) + (remainder).

For the polynomial example you can write

$$\frac{6x^3 - 13x^2 + 4x + 5}{2x + 1} = (3x^2 - 8x + 6) + \frac{^{-}1}{2x + 1}$$

$6x^3 – 13x^2 + 4x + 5 = (2x + 1)(3x^2 – 8x + 6) + (^{-}1)$

Which is of the same format as the numerical example:

(dividend) = (divisor) $\times$ (quotient) + (remainder).

The zero of the divisor is $^{-}\frac{1}{2}$, i.e., $2x + 1 = 0$ for $x = ^{-}\frac{1}{2}$.

Verify that $P(^{-}\frac{1}{2}) = ^{-}1$. Is this a coincidence, that you get the remainder? That you will investigate further in the next self mark exercise.

Self mark exercise 10

1. Find quotient and remainder when $P(x)$ is divided by $D(x)$ where
   a) $P(x) = x^2 – 3x + 4$ and $D(x) = x + 4$
   b) $P(x) = 3x^3 – 13x^2 – 20$ and $D(x) = x – 5$
   c) $P(x) = 4x^3 + 6x^2 – 2x – 1$ and $D(x) = 2x + 1$
2. Divide $P(x)$ by $D(x)$ and find the remainder. Calculate also the value of $P(x)$ for the value of making the divisor zero. Start by entering the results already obtained.

   Continued on next page.

Self mark exercise 10 continued.

Complete the following table:

$P(x)$	$D(x) = ax + b$	Remainder	$P(-\frac{b}{a})$
$6x^3 - 13x^2 + 4x + 5$	$2x + 1$	-1	$P(-\frac{1}{2}) = -1$
$x^2 - 3x + 4$	$x + 4$		
$3x^3 - 13x^2 - 20$	$x - 5$		
$4x^3 + 6x^2 - 2x - 1$	$2x + 1$		
$6x^2 - 5x - 3$	$x - 1$		
$x^3 - 10$	$x - 2$		
$4x^3 - 13x^2 + 3x - 3$	$4x - 1$		

3. Can you make a conjecture about the remainder when a polynomial $P(x)$ is divided by $(ax + b)$?

Check your answers at the end of this unit.

Section C2: Remainder theorem

The conjecture you were to come up with in the last exercise is not so hard to prove.

Let us look at a simpler case first: dividing a polynomial $P(x)$ by $(x - a)$. Remember that for all division the identity

(dividend) = (divisor) × (quotient) + (remainder) holds.

If $P(x)$ is divided by $(x - a)$ and the quotient is $Q(x)$ and the remainder R (a constant, a polynomial of degree 0) you have the identity

$P(x) = (x - a)\ Q(x) + R$

Because this is an identity (an equation holding for ALL values of the variable x) we can take any value for x, for example also the value $x = a$. This gives:

$$P(a) = (a - \mathrm{a})\ Q(a) + R$$

$$P(a) = 0 + R$$

$$P(a) = R$$

In words: The remainder when the polynomial $P(x)$ is divided by $(x - a)$ is $P(a)$. This is called the **remainder theorem.**

The theorem can also be written down in case you divided the polynomial $P(x)$ by the factor $(ax + b)$. In this case you have the identity

$P(x) = (ax + b)\ Q(x) + R$

Taking for x the value $-\dfrac{b}{a}$

$$P\left(\frac{-b}{a}\right) = (a \times \frac{b}{a} - b)\ Q\left(\frac{-b}{a}\right) + R$$

$$P\left(\frac{-b}{a}\right) = 0 + R$$

$$P\left(\frac{-b}{a}\right) = R.$$

In words: The remainder when the polynomial $P(x)$ is divided by $(ax + b)$ is $P\left(\frac{-b}{a}\right)$.

Self mark exercise 11

1. Find the remainder when

   a) $x^3 - 6x^2 + 4$ is divided by $x + 1$

   b) $x^4 - 8x^2 + 3$ is divided by $x - 1$

   c) $x^{15} - 1$ is divided by $x + 1$

   d) $4x^3 - 5x^2 + x$ is divided by $x - 2$

2. Find the value of k

   a) when $2x^4 + kx^3 + 8$ is divided by $x + 2$ the remainder is 16

   b) when $kx^3 - 4x^2 + 6$ is divided by $x - 3$ the remainder is $^{-}3$

   c) when $2x^5 + kx^3 + x$ is divided by $x + 1$ the remainder is $^{-}5$

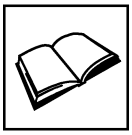
3. The remainder when $2x^3 + ax^2 + bx - 2$ is divided by $x + 2$ is 4. When divided by $x - 2$ the remainder is 24. Find the values of a and b.

4. If the polynomial $P(x)$ is divided by $(x - a)(x - b)$ i.e. a quadratic expression, the remainder will be a linear expression of the form $Ax + B$. If the quotient is $Q(x)$ the identity

   $P(x) = (x - a)(x - b)\ Q(x) + (Ax + B)$ will hold.

   Find the remainder of $Ax + B$.

Check your answers at the end of this unit.



Section C3: Factor theorem

If a polynomial $P(x)$ is divided by $(x - a)$ you saw that the identity

$P(x) = (x - a)\ Q(x) + R$ holds.

In case that the remainder $R = 0$, $x - a$ is a factor of $P(x)$. This is the **factor theorem**:

$(x - a)$ is a factor of the polynomial $P(x)$ if $P(a) = 0$

or also:

$(ax + b)$ is a factor of the polynomial $P(x)$ if $P\left(\frac{-b}{a}\right) = 0$.

The factor theorem is used to find rational factors of polynomials.

Example:

Find the rational factors of $P(x) = 2x^3 + x^2 - 13x + 6$.

Try factors of the constant 6: $\pm 1, \pm 2, \pm 3, \pm 6$.

$P(1) = 2 + 1 - 13 + 6 \neq 0$, so $(x - 1)$ is not a factor.

$P(^-1) = {}^-2 + 1 + 13 + 6 \neq 0$, so $(x + 1)$ is not a factor.

$P(2) = 16 + 4 - 26 + 6 = 0$, so $(x - 2)$ is a factor.

$P(x) = (x - 2)\ (2x^2 + 5x - 3)$ by long division.

$P(x) = (x - 2)\ (2x - 1)(x + 3)$ by factorising the quadratic.

Note that the factorisation now gives an easy way to solve the equation $2x^3 + x^2 - 13x + 6 = 0$.

$$2x^3 + x^2 - 13x + 6 = 0$$

$$(x - 2)\ (2x - 1)(x + 3) = 0$$

$$x - 2 = 0 \text{ or } 2x - 1 = 0 \text{ or } x + 3 = 0$$

$$x = 2 \text{ or } x = \frac{1}{2} \text{ or } x = {}^-3$$

Self mark exercise 12

1. Find the value of x if

   a) $x - 2$ is a factor of $3x^3 + kx - 20$

   b) $x + 1$ is a factor of $kx^3 + 8x^2 + 3x - 2$

2. Find the value of p and q if

   a) $x + 1$ and $x - 2$ are factors of $x^3 + px^2 + qx + 6$

   b) $x + 2$ and $x - 3$ are factors of $px^3 - x^2 - 13x + q$

3. Factorise the polynomials:

   a) $2x^3 + 5x^2 + x - 2$　　b) $3x^3 - x^2 - 38x - 24$

   c) $x^3 - 8$　　d) $x^3 + 27$

   e) $x^3 - a^3$　　f) $x^3 + a^3$

4. a) Factorise $5x^2 + 48x + 27$. Use tour factorisation to write 54827 as the product of two prime numbers.

   b) What expression would you factorise to express 41309 as the product of two prime numbers? Factorise the expression and find the prime numbers.

Check your answers at the end of this unit.

You have come to the end of this module. I hope you found it useful, found some enjoyment in going through the module, but above all that it has helped you to improve and renovate your classroom practice.

You should have strengthened your knowledge of pupils' difficulties in learning basic algebra and how these difficulties might be avoided by using concrete materials, precise mathematical language and investigative methods.

You should have gained confidence in creating a learning environment for your pupils in which they can

(i) acquire with understanding knowledge on expansion and factorisation of algebraic expressions

(ii) illustrate with paper models algebraic identities

(iii) investigate arithmetical patterns leading to generalisations expressed in algebraic form.

(iv) investigate and resolve algebraic fallacies

You may have extended your knowledge in algebra having acquired knowledge on the binomial theorem, Pascal's triangle and pyramid, remainder and factor theorem.

Module 5, Practice activity

1. a) List the knowledge and skills you expect pupils in junior secondary school to acquire through learning simplification, expansion and factorisation.

   b) Describe how you create opportunities in your class for pupils to acquire the knowledge and skills you listed in 1a.

2. a) Write down any three digits, for example 653
   Reverse the digits (356) and subtract the two numbers (smaller from the larger 653 – 356 = 297)
   Add to this answer (297) the number obtained by reversing the digits in the answer (792).
   Compare with others. Try other numbers. Can you prove the result using algebra? Try four digit numbers and investigate.

   b) Write down a three digit number with the digits different from each other. Add together all the (6) different two digits numbers you can form from the three digit number. Divide your answer by the sum of the digits of the original number.

   Compare with others. Try other numbers. Can you prove the result using algebra?

   c) Set the above investigations to your pupils and write an evaluation.

3. a) Make a worksheet for the pupils with number patterns, which when generalised will lead to the expression:

   (i) $(a + b)^2 = a^2 + 2ab + b^2$
   (ii) $(a - b)^2 = a^2 - 2ab + b^2$
   (iii) $a - (b - c) = a - b + c$
   (iv) $a - (b + c) = a - b - c$

   b) Try out the worksheet(s) with your pupils.

   c) Write a critical evaluation of the activity.

4. a) Design a game to consolidate factorisation.

   b) Try out your game with your pupils.

   c) Write an evaluative report.

5. Have a look at the content of unit 4. Could some of the material be presented to high achievers in form 3? or used in a mathematics club? Justify your answer. If your answer is positive try out the activity you feel (some) pupils can do and write an evaluative report.

6. Did this module lead to any changes in the methods you use to assist pupils in the learning of algebra?

   If yes, list the changes and explain why you decided to make a change.

   If no, explain why.

Present your assignment to your supervisor or study group for discussion.

Summary

The key to learning algebra at the ages of 11-15 is **obvious usefulness** facilitated by **working models**. The authors hope that your teaching will supply that key!

Unit 4: Answers to self mark exercises

Self mark exercise 1

1.

	Degree	Coefficient of x^3
a)	3	2
b)	3	5
c)	4	0
d)	0	0
e)	8	$^{-}1$

2 a) $6x^4 - 7x^3 + 2x^2 + 9x - 6$

b) $4x^4 + 13x^3 - 3x^2 - 2x + 6$

c) $6x^{12} - x^8 - 8x^6 - 12x^4 - 5x^2 + 2$

d) $8x^4 - 18x^3 + 13x^2 + 24x - 45$

3 a) $(n + m)$

b) $2n$

c) tn

Self mark exercise 2

1. a) $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

b) $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

c)

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

2.

1 6 15 20 15 6 1

↘ ↙ ↘ ↙ ↘ ↙ ↘ ↙ ↘ ↙ ↘ ↙

1 7 21 35 35 21 7 1

Adding two numbers next to each other in, say row 6, gives the numbers for row 7. Add a starting and finishing 1 to row 7.

Self mark exercise 3

1. a) $x^7 + \frac{7}{1}x^6 + \frac{7\times 6}{1\times 2}x^5 + \frac{7\times 6\times 5}{1\times 2\times 3}x^4 + \frac{7\times 6\times 5\times 4}{1\times 2\times 3\times 4}x^3 +$

$\frac{7\times 6\times 5\times 4\times 3}{1\times 2\times 3\times 4\times 5}x^2 + \frac{7\times 6\times 5\times 4\times 3\times 2}{1\times 2\times 3\times 4\times 5\times 6}x + 1$

b) $x^8 + \frac{8}{1}x^7 + \frac{8\times 7}{1\times 2}x^6 + \frac{8\times 7\times 6}{1\times 2\times 3}x^5 + \frac{8\times 7\times 6\times 5}{1\times 2\times 3\times 4}x^4 +$

$\frac{8\times 7\times 6\times 5\times 4}{1\times 2\times 3\times 4\times 5}x^3 + \frac{8\times 7\times 6\times 5\times 4\times 3}{1\times 2\times 3\times 4\times 5\times 6}x^2 +$

$\frac{8\times 7\times 6\times 5\times 4\times 3\times 2}{1\times 2\times 3\times 4\times 5\times 6\times 7}x + 1$

2. $(1+x)^n = 1 + nx + \frac{n\times(n-1)}{1\times 2}x^2 + \frac{n\times(n-1)\times(n-2)}{1\times 2\times 3}x^3 +$

$\frac{n\times(n-1)\times(n-2)\times(n-3)}{1\times 2\times 3\times 4}x^4 + \ldots + \ldots\ldots +$

$\frac{n\times(n-1)\times(n-2)\times(n-3)\times(n-4)\times\ldots\times(n-\{r-1\})}{1\times 2\times 3\times 4\times 5\times\ldots\times r}x^r + \ldots + \ldots \quad x^n$

Self mark exercise 4

1. $32x^4 + 32x^3 + 24x^2 + 8x + 1$
2. $^-8x^3 + 12x^2 - 6x + 1$
3. $\frac{1}{32}x^5 + \frac{5}{16}x^4 + \frac{5}{4}x^3 + \frac{5}{2}x^2 + \frac{5}{2}x + 1$
4. $16x^4 - 96x3y + 216x2y^2 - 216xy^3 + 81y^4$
5. $x^6 + 3x^4y^2 + 3x2y^4 + y^6$
6. $\frac{1}{32}x^5 - \frac{5}{48}x^4y + \frac{5}{36}x^3y^2 - \frac{5}{54}x^2y^3 + \frac{5}{162}xy^4 - \frac{1}{243}y^5$
7. 358 213.55
8. $1 + 5x + 5x^2 - 10x^3 + \ldots$
9. a) $^-3\ 421\ 440$ b) $^-672\ 000$
10. 258 660 864
11. a) 168.439 b) 123.587
12. ab) 56 cd) 495 ef) 5 200 300 g) $\binom{n}{r} = \binom{n}{n-r}$

h) $\binom{n}{r} = \frac{n!}{r!\times(n-r)!}$ *and* $\binom{n}{n-r} = \frac{n!}{(n-r)!\ \{n-(n-r)\}!} = \frac{n!}{(n-r)!\times r!}$

Self mark exercise 5

1. convergent
2. convergent
3. convergent

Self mark exercise 6

1. $1+\frac{1}{5}x-\frac{2}{25}x^2+\frac{6}{125}x^3-...$ 1.0040
2. $1+\frac{5}{2}x-\frac{25}{8}x^2+\frac{125}{16}x^3-...$ 10.2470
3. $1-\frac{4}{3}x-\frac{16}{9}x^2-\frac{320}{81}x^3-\ ...$ 2.4662
4. $1-\frac{3}{2}x-\frac{9}{8}x^2-\frac{27}{16}x^3-...$ 3.1623
5. $1-3x+\frac{3}{2}x^2+\frac{1}{2}x^3+\frac{3}{8}x^4+..$ 1.7320
6. $\sqrt{2}\ \{1+\frac{1}{4}x-\frac{1}{32}x^2+\frac{1}{128}x^3+...)$
7. a) $1-x+x^2-x^3+x^4$ ... convergent for $^-1 < x < 1$

   b) $1-2x+3x^2-4x^3+5x^5$ ... convergent for $^-1 < x < 1$

   c) $1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4-...$ convergent for $^-1 < x < 1$
8. $1-3x+6x^2-10x^3+15x^4-....$ 1.0306
9. a) $1-6x+27x^2-108x^3+405x^4-\ ....$

   b) $1+20x+240x^2+2240x^3+17920x^4+...$

   c) $\frac{1}{4}+\frac{1}{4}x+\frac{3}{16}x^2+\frac{1}{8}x^3+\frac{5}{64}x^4+...$
10. a) $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-....$

    b) $1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+....$

Problem Solving 1

1. Sum in nth row 2^{n-1}.
2. Sum is one less than the number the bottom corner of the rectangle points to.

3. 252

4. $2^{n-1} + 2n - 3$

5. 20

Self mark exercise 7

1. a) $(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

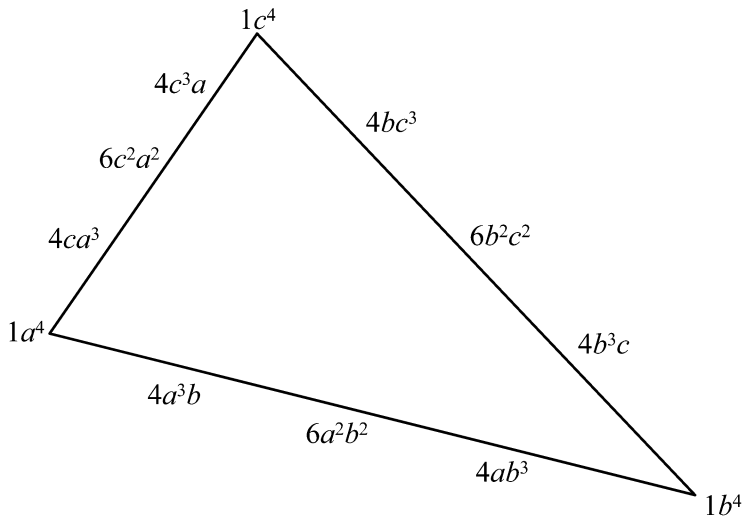
   b) $(a + b + c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3b^2c + 3bc^2 + c^3 + 3c^2a + 3ca^2 + 6abc$

   c) $(a + b + c)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4 + + 4c3a + 6c^2a^2 + 4ca^3 + 12a^2bc + 12ab^2c + 12abc^2$.

Self mark exercise 8

1.

```
            1
         4     4
      6     12     6
   4    12     12    4
1     4      6     4     1
```



2. The numbers along the sides of the triangular section of the pyramid are exactly as in Pascal’s triangle. In the above example along the three sides you find the numbers 1 4 6 4 1 corresponding to the fifth row in Pascal’s triangle i.e. the coefficients in the expansion of $(x + y)^4$.

Self mark exercise 9

1.

1	x1	1
1 1	x5	5 5
1 2 1	x10	10 20 10
1 3 3 1	x10	10 30 30 10
1 4 6 4 1	x5	5 20 30 20 5
1 5 10 10 5 1	x1	1 5 10 10 5 1

2. a) $(a + b + c)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 + 5b4x +$
$10b^3c^2 + 10b^2c^3 + 5bc^4 + c^5 + 5c^4a + 10c3a^2 + 10c^2a^3 + 5ca^4 +$
$20a^3bc + 20ab^3c + 20abc^3 + 30a^2b^2c + 30ab^2c^2 + 30a^2bc^2$

   b) $(a + 3b + 2c)^3 = a^3 + 9a^2b + 27ab^2 + 27b^3 + 54b^2c + 36bc^2 + 8c^3 +$
$12c^2a + 6ca^2 + 36abc$

   c) $(a + 2b + c)^4 = a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4 + 32b^3c +$
$24b^2c^2 + 8bc^3 + c^4 + 4c^3a + 6c^2a^2 + 4ca^3 + 24a^2bc + 48ab^2c +$
$24abc^2$

3. The coefficient of $a^pb^qc^r$ in the expansion of $(a + b + c)n$ where
$p + q + r = n$ is $\dfrac{n!}{p!\ q!\ r!}$. This is also written as $\begin{pmatrix} n \\ p,\ q,\ r \end{pmatrix}$

   For example the coefficient ab^2c^2 in the expansion of $(a + b + c)^5$ is

$$\begin{pmatrix} 5 \\ 1,\ 2,\ 2 \end{pmatrix} = \frac{5!}{1!\ 2!\ 2!} = \frac{1\,.\,2\,.\,3\,.\,4.\,.5}{1.\,2.\,.1.\,.2} = 30$$

4. The general term in the expansion will be

$$\begin{pmatrix} n \\ n_1,\ n_2,\ n_3,\ \ldots\ldots,\ n_r \end{pmatrix} a_1^{n_1} \cdot a_2^{n_2}\ .\ a_3^{n_3}\ .\ \ldots\ldots\ \cdot\ a_r^{n_r}$$

Self mark exercise 10

1. a) Quotient $(x - 7)$, remainder 32

   b) Quotient $(3x^2 + 2x + 10)$, remainder 30

   c) Quotient $2x^2 + 2x - 2$, remainder 1

2. Remainder = $P\left(\frac{-b}{a}\right)$

$^{-}1$

32

30

1

$^{-}2$

$^{-}18$

$^{-}3$

3. The remainder when $P(x)$ is divided by $(ax + b)$ is $P(^{-}\frac{b}{a})$

Self mark exercise 11

1. a) $^{-}3$ b) $^{-}1$ c) $^{-}2$ d) 14
2. a) 3 b) 1 c) 2
3. $a = 4$, $b = {}^{-}3$
4. $\frac{f(a) - f(b)}{a - b}x + \frac{af(b) - bf(a)}{a - b}$

Self mark exercise 12

1. a) $^{-}2$ b) 3
2. a) $p = {}^{-}4$, $q = {}^{-}1$ b) $p = 2$, $q = {}^{-}6$
3. a) $(x + 1)(x + 2)(2x - 1)$ b) $(x + 3)(x - 4)(3x + 2)$

   c) $(x + 2)(x^2 - 2x + 4)$ d) $(x - 3)(x^2 + 3x + 9)$

   e) $(x + a)(x^2 - ax + a^2)$ f) $(x - a)(x^2 + ax + a^2)$
4. a) $(5x + 3)(x + 9)$; 503, 109

   b) $4x^2 + 13x + 9 = (4x + 9)(x + 1)$; 409, 101

References

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Hart, K. *Children's Understanding of Mathematics 11 - 16*, 1981, ISBN 071 953 772x

Additional References

In preparing the materials included in this module we have borrowed ideas extensively from other sources and in some cases used activities almost intact as examples of good practice. As we have been using several of the ideas, included in this module, in teacher training the original source of the ideas cannot be traced in some cases. The main sources are listed below.

Boyer, C.B., 1991, *A History of Mathematics (2nd Edition)*, ISBN 047 154 3977

Eagle M, *Exploring Mathematics through history*, 1995, ISBN 052 145 6266

NCTM, *Algebra in a Technological World*, 1995, ISBN 087 353 3267

NCTM, *Learning and Teaching Algebra*, 1984, ISBN 087 353 2686

NCTM, *Patterns and Functions, Addenda Series*, 1994, ISBN 087 353 3240

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Open University, *Preparing to Teach Equations*, PM 753C,

ISBN 033 517 4396

Reimer W, *Historical Connections in Mathematics Volume 1, 2 and 3*, 1992, ISBN 188 143 1355, ISBN 188 143 138*x*, ISBN 188 143 1495

The Penguin Dictionary of Mathematics, 1989, ISBN 014 051 1199

Tauris, I.B., 1991, *The Crest of the Peacock. Non European Roots of Mathematics*, ISBN 185 043 2856

UB-INSET, *Algebra for Everyone*, 1996, University of Botswana

Further reading

The Maths in Action book series are for use by pupils in the classroom. The books are using a constructivist, activity based approach, including problem solving, investigations, games and challenges in line with the ideas in this module.

OUP/Educational book Service, Gaborone, *Maths in Action Book 1* ISBN 019 571776 7

OUP/Educational book Service, Gaborone, *Maths in Action Book 2*

OUP/Educational book Service, Gaborone, *Maths in Action Teacher's File Book 1*

OUP/Educational book Service, Gaborone, *Maths in Action Teacher's File Book 2*

G Glossary

Algebra	the branch of mathematics that deals with the general properties of numbers and generalisations of relationships among numbers
Algebraic expression	any combination of numbers and variables and algebraic operations
Binomial	a polynomial* consisting of two terms e.g. $2a + b$, $2p^3 - 2q^2$
Common factor	a number and/or expression that divides two or more given numbers or algebraic expressions exactly
Degree of polynomial	highest index of the variable x with non zero coefficient (for polynomials in one variable)
Dividend	a number or polynomial that is divided by another number or polynomial
Divisor	a number or polynomial that divides another number or polynomial
Expansion	writing an algebraic expression as the sum of a number of terms e.g. $(x + 3)(x + 4) = x^2 + 7x + 12$
Factor	a number or polynomial that divides a given number or polynomial exactly
Factorial n	written n! represent the product of all the integers 1 to n inclusive, i.e., $1 \times 2 \times 3 \times 4 \times ... \times n$. By convention 0! = 1
Factorising	writing a given sum of terms as a product of factors, i.e., $x^2 + 7x + 12 = (x + 3)(x + 4)$
Factor theorem	$(x - a)$ is a factor of a polynomial $P(x)$ if $P(a) = 0$
Fallacy (algebraic)	results obtained through algebra that appear true but are actually false
HCF	highest common factor: the largest number or algebraic expression with the highest exponents that is a common factor of two or more given numbers or algebraic expressions
Monomial	expression of the form ax^n or $ax^n y^m$ (or with more variables to an index) where a is a real number and n and m are non negative integers
Polynomial	sum of monomials. For one variable the general form is $a_0 + a_1x + a_2x^2 + ... + a_nx^n$

Quotient the result of dividing one number or polynomial by another

Remainder a number or polynomial remaining after one number or polynomial is divided into another

Remainder theorem if the polynomial $P(x)$ is divided by $(x - a)$ the remainder is equal to $P(a)$.

Variable representation of a changing quantity

Appendix 1

Muhammad ibn Musa al-Khwarizmi (c780 - c 850)

Arab mathematician from Khiva, now part of the Uzbek Republic of the USSR. In his *Al-jam' w'al-tafriq ib hisab al-hind* (Addition and Subtraction in Indian Arithmetic), al-Khwarizmi introduced the Indian system of numerals to the West. He also wrote a treatise on algebra, *Hisab al-jabr w'al-muqabala* (Calculation by Restoration and Reduction); from 'al-jabr' comes the word 'algebra'. From al-Khwarizmi's name was derived the term 'algorism' (referring originally to the Hindu-Arabic number system, but later to computation in a wider sense), from which in turn comes 'algorithm'.

From: *The Penguin Dictionary of Mathematics*, 1989, page 14.

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Al-Khwarizmi

From an intellectual viewpoint, the ninth century A.D. was essentially a Moslem century. The activity of the scholars of Islam was far superior to that of any other group. Flourishing under Caliph al-Mamum, the Moslem mathematician, astronomer, and geographer Mohammed ibn-Musa al-Khwarizmi influenced mathematical thought more than did any other medieval writer on the subject. He is generally known for his work in algebra and astronomy and for his mathematical tables; but perhaps one of the greatest achievements was his introduction of the so-called Hindu-Arabic notational system to the Western world. His writings were the main channel by which the "new" numerals became known in the West, and through them he revolutionised the common processes of computation.

Al-Khwarizmi's best-known book, *Hisab al-jabr w'al-muqabalah* ("science of restoration [or reunion] and opposition"), was written to show that "what is easiest and most useful in arithmetic such as men constantly require in cases of inheritance, legacies, partition, lawsuits and trade, and in all their dealings with one another." This work, containing 79 pages on inheritance cases, 16 pages of measurement problems, and 70 pages of algebra, begins with a brief recapitulation of the place-value number system based on ten and then goes on to teach the use of these positional numerals in computation.

There is, for example, a systematic treatment of surds. Al-Khwarizmi noted that, to double a root, one must multiply the square by four; to triple it, by nine; and to halve it, by one-fourth. He gave examples of the multiplication and division of surds, concluded the series by saying , "Likewise with other numbers," thus indicating that the reader was to deduce a general rule from the examples given. He used this same technique in mercantile problems, where many of the examples imply a use of proportionality. "If ten items cost six [coins], what will four cost?" After seeing how several such problems are solved, one is to do "likewise with other numbers."

It should be noted that throughout the treatise numbers are expressed in words, not symbols, numerals being used only in diagrams and a few marginal notes. The methods of computation thus taught verbally could be applied only to the then new place-value system of numeration used by the Moslems. Thus the promulgation of this work of al-Khwarizmi gave strong impetus to the scholars of the West to learn what we now call the numerals of the Hindu-Arabic system and to become skilful in manipulating them.

From: NCTM *Historical Topics for the Mathematics Classroom*, 1989, page 76 - 77

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Besides astronomical tables, and treatises on the astrolabe and the sundial, al-Khwarizmi wrote two books on arithmetic and algebra which played very important roles in the history of mathematics. One of these survives only in a unique copy of a Latin translation with the title *De numero indorum (Concerning the Hindu Art of Reckoning)*, the original Arabic version having since been lost. In this work, based presumably on an Arabic translation of Brahmagupta, al-Khwarizmi gave so full an account of the Hindu numerals that he probably is responsible for the wide-spread but false impression that our system of numeration is Arabic in origin. Al-Khwarizmi made no claim to originality in connection with the system, the Hindu source of which he assumed as a matter of course; but when subsequent Latin translations of his work appeared in Europe, careless readers began to attribute not only the book but also the numeration to the author. The new notation came to be known as that of al-Khwarizmi, or more carelessly, algorismi; ultimately the scheme of numeration making use of the Hindu numerals came to be called simply algorism or algorithm, a word that, originally derived from the name al-Khwarizmi, now means, more generally, any peculiar rule of procedure or operation—such as Euclidean method for finding the greatest common divisor.

Through his arithmetic, al-Khwarizmi's name has become a common English word; through the title of his most important book, *Al-jabr wa'l muqabalah*, he has supplied us with an even more popular term. From this title has come the word *algebra*, for it is from this book that Europe later learned the branch of mathematics bearing this name. Diophantus sometimes is called "the father of algebra," but this title more appropriately belongs to al-Khwarizmi. It is true that in two respects the work of al-Khwarizmi represented a retrogression from that of Diophantus, First, it is on a far more elementary level that that found in the Diophantine problems and, second, the algebra of al-Khwarizmi is thoroughly rhetorical, with none of the syncopation found in the Greek *Arithmetica* or in Brahmagupta's work. Even numbers were written out in words rather than symbols! It is quite unlikely that al-Khwarizmi knew of the work of Diophantus, but he must have been familiar with at least the astronomical and computational portions of Brahmagupta; yet neither al-Khwarizmi nor other Arabic scholars made use of syncopation (syncopation) or of negative numbers. Nevertheless, the *Al-jabr* comes closer to the elementary algebra of today than the works of either Diophantus or Brahmagupta, for the book is not concerned with difficult problems in indeterminate analysis but with a straightforward and elementary exposition of the solution of equations, especially of second degree. The Arabs in general loved a good clear argument from premise to conclusion, as well as systematic organization—respects in which neither Diophantus nor the Hindus excelled. The Hindus were strong in association and analogy, in intuition and an aesthetic and imaginative flair, whereas the Arabs were more practical-minded and down-to-earth in their approach to mathematics.

The *Al-jabr* has come down to us in two versions, Latin and Arabic, but in the Latin translation, *Libber algebra et almucubala*, a considerable portion of the Arabic draft is missing. The Latin, for example, has no preface, perhaps because the author's preface in Arabic gave fulsome praise to Mohammed, the prophet, and to al-Mamum, "the Commander of the Faithful." Al-Khwarizmi wrote that the latter had encouraged him to compose a short work on "Calculating by (the rules of) Completion and Reduction, confining it to what is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partitions, law-suits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned" [Karpinski, 1915, p 15]

It is not certain just what the terms *al-jabr* and *muqabalah* mean, but the usual interpretation is similar to that implied in the translation above. The word *al-jabr* presumably meant something like "restoration" or "completion" and seems to refer to the transposition of subtracted terms to the other side in an equation; the word *muqabalah* is said to refer to "reduction" or "balancing"—that is, the cancellation of like terms on opposite sides of the equation. Arabic influence in Spain long after the time of al-Khwarizmi is found in *Don Quixote*, where the word *algebrista* is used for a bone-setter, that is, a "restorer."

The Latin translation of al-Khwarizmi's *Algebra* opens with a brief introductory statement of the positional system for numbers and then proceeds to the solution, in six short chapters, of the six types of equations made up of the three kinds of quantities: roots, squares, and numbers (that is x, x^2, and numbers). Chapter I, in three short paragraphs, covers the case of squares equal to roots, expressed in modern notation as $x^2 = 5x$, $\frac{x}{3} = 4x$, and $5x^2 = 10x$, giving the answers $x = 5$, $x = 12$ and $x = 2$ respectively. (The root $x = 0$ was not recognised.) Chapter II covers the case of squares equal to numbers, and Chapter III solves the case of roots equal to numbers, again with three illustrations per chapter to cover the cases in which the coefficient of the variable term is equal to, more than, or less than one. Chapter IV, V and VI are more interesting, for they cover in turn three classical cases of three-term quadratic equations: (1) squares and roots equal to numbers, (2) squares and numbers equal to roots, and (3) roots and numbers equal to squares. The solutions are "cookbook" rules for "completing the square" applied to specific instances. In each case only the positive answer is given.

So systematic and exhaustive was al-Khwarizmi's exposition that the readers must have had little difficulty in mastering the solutions. In this sense, then, al-Khwarizmi is entitled to be known as "the father of algebra." However, no branch of mathematics springs up fully grown, and we cannot help but ask where the inspiration for Arabic algebra came from. To this question no categorical answer can be given; but the arbitrariness of the rules and the strictly numerical form of the six chapters remind us of ancient Babylonian and medieval Indian mathematics. The exclusion of indeterminate analysis, a favourite Hindu topic, and the avoidance of any syncopation, such as found in Brahmagupta, might suggest Mesopotamia as more likely a source than India. As we read beyond the sixth chapter, however, an entirely new light is thrown on the question. Al-Khwarizmi continued:

"We have said enough so far as numbers are concerned, about the six types of equations. Now, however, it is necessary that we should demonstrate geometrically the truth of the same problems which we have explained in numbers."

The ring in this passage is obviously Greek rather than Babylonian or Indian. There are, therefore, three main schools of thought on the origin of Arabic algebra: one emphasises Hindu influences, another stresses the Mesopotamian, or Syriac-Persian, tradition and the third points to Greek inspiration. The truth is probably approached if we combine the three theories. The philosophers of Islam admired Aristotle to the point of aping him, but eclectic Mohammedan mathematicians seem to have chosen appropriate elements from various sources.

The *Algebra* of al-Khwarizmi betrays unmistakable Hellenic elements, but the first geometric demonstrations have little in common with classical Greek mathematics.

[Extracts from Carl B. Boyer *A History of Mathematics* (2nd Edition) p 227 - 232]

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Abu Jafar Muhammad ibn Musa al-Khwarizmi (to give him his full name, which means Muhammad, the father of Jafar and the son of Musa, from Khwarizm) was born in about 780. The name 'al-Khwarismi' suggest that either he or his family came from Khwarizm, east of the Caspian Sea in what is today Soviet Central Asia. Little is known of his early life. There is a reference to al-Khwarizmi as 'al-Majusi' in a book entitled *History of Kings and Envoys* by al-Tabari an Arab historian of the ninth century. In the Pahlavi language, a Zoroastrian was sometimes referred to as 'magus', from which comes our words 'magi' and 'magician'. There is therefore, the view that al'Khwarizmi may have been of Zoroastrian heritage and had acquired his early knowledge of Indian mathematics and astronomy from Zoroastrian clergy, some of whom were reputed to be well acquainted with these subjects.

In about 820, after establishing a reputation as a talented scientist in Merv, the capital of the Eastern provinces of the Abbasid caliphate, he was invited by Caliph al-Mamum to move to Baghdad, where he was appointed first astronomer and then head of the library at the House of Wisdom. He continued to serve other caliphs, including al-Wathiq during his short rule from 842 to 847. There is a story, told by the historian al-Tabari, that when

Caliph al-Wathiq lay seriously ill he asked al-Khwarizmi to cast his horoscope and find out whether he would live. Al-Khwarizmi assured the caliph that he would live another fifty years, but al-Wathiq died within ten days. Whether this story illustrates al-Khwarizmi's highly developed sense of survival or his ineptness as a fortune teller, it is difficult to say. We know little else of his later life except that he died in about 850.

Al-Khwarizmi also constructed a *zij* (a set of astronomical tables) that was to remain important in astronomy for the next five centuries. The origin of this *zij* is interesting, for it shows how Indian mathematics and astronomy entered the Arab world directly for the first time. An Arab historian al -Qifti (c 1270) reported that in the year AH 156 (AD 773) a man well versed in astronomy, by the name of Kanaka, came to Baghdad as a member of a diplomatic mission from Sind, in northern India. He brought with him Indian astronomical texts, including *Surya Siddhanta* and the works of Brahmagupta. Caliph al-Mansur ordered that some of these texts should be translated into Arabic and, according to the principles given in them, that a handbook be constructed for use by Arab astronomers. The task was delegated to al-Fazari, who produced a text which came to be known by later astronomers as the *Great Siindhind*. The word *sinshind*is derived from the Sanskrit word *siddhanta*, meaning an astronomical text. It was mainly on the bias of this text, as well as some other elements from Babylonian and Ptolemaic astronomy, that al-Khwarizmi constructed his *zij*. Unfortunately, the original Arabic text is no longer extant. But a Latin translation, made in 1126 from the edited version produced by Maslama al-Majriti (a Spanish astronomer who lived in Cordoba in about the year 1000), became one of the most influential astronomical texts in medieval Europe.

[Extract from: Tauris, I.B., 1991, *The Crest of the Peacock*, page 304 - 306]

Appendix 2

Diophantus of Alexandria (c AD 250)

Greek mathematician and the author of the *Arithmetica*, of which ten of the original books are extant. About 130 problems are considered, some of which are surprisingly hard, in the field of what have since become known as Diophantine equations.

Diophantine equation

Any equation usually in several unknowns, that is studied in a problem whose solutions are required to be integers, or sometimes more general rational numbers. Examples of such problems are

(1) To find integers x, y that satisfy $11x + 3y = 1$.

(2) To find rational numbers x, y, z such that $x^3 + y^3 = z^3$

From: *The Penguin Dictionary of Mathematics*, 1989, page 99.

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The Greek Diophantus (c A.D. 75 or 250) author of the classic *Arithmetica*, also wrote a treatise on polygonal numbers, only a fragment of which is known. In it he generalises Statement 4 above. [Eight times any triangular number plus 1 is a square number]. He also solves this problem: Given a number, find in how many ways it can be a polygonal number. (Note, in the table given earlier, that 55 is both a triangular and a heptagonal number; 81 is both a square and heptagonal.) The manuscript breaks off in the middle of this problem.

From: NCTM *Historical Topics for the Mathematics Classroom*, 1989, page 57

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For some common fractions the Greeks had special symbols, but for others the fractions were written with one accent on the numerator and two on the denominator, which was written twice. Thus we have ιγ‘κθ“κθ“ = $\frac{13}{20}$.

Diophantus sometimes used a form similar to ours today, but he wrote the denominator above the numerator and without the horizontal line.

From: NCTM *Historical Topics for the Mathematics Classroom*, 1989, page 96

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Diophantus introduced the syncopated style of writing equations. He studied and worked at the University of Alexandria, Egypt, where Euclid had once taught. Very little else is known about his life except what is claimed in the following algebraic puzzle rhyme from the *Anthologia Palatine*:

> "Here lies Diophantus." The wonder behold—

Through art algebraic, the stone tells old:

"God gave him his boyhood one-sixth of his life,

One-twelfth more as youth while whiskers grew rife;

And yet one-seventh ere marriage begun;

In five years there came a bouncing new son.

Alas, the dear child of master and sage

Met fate at just half his dad's final age.

Four years yet his studies gave solace from grief;

Then leaving scenes earthly he, too, found relief.

(What answer do you get for Diaophantus' age? Eighty-four?)

Diophantus' claim to fame rest on his *Arithmetica*, in which he gives an ingenious treatment of indeterminate equations—usually two or more equations in several variables that have an infinite number of rational solutions. These are often called Diophantine equations, although he was not the first to solve such systems. His approach is clever, but he does not envelop a systematic method for finding general solutions. His approach is along Babylonian lines in the sense that he expresses all unknowns in terms of a parameter and then obtains an equation containing only the parameter.

From: NCTM *Historical Topics for the Mathematics Classroom*, 1989, page 240

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At the beginning of this period [the "Silver Age" about AD 250 to 350], also known as the Later Alexandrian Age, we find the leading Greek algebraist, Diophantus of Alexandria, and towards its close there appeared the last significant Greek geometer, Pappus of Alexandria. No other city has been the centre of mathematical activity for so long a period as was Alexandria from the days of Euclid (ca. 300 BC) to the time of Hypatia († AD 415). It was a very cosmopolitan centre, and the mathematics that resulted from Alexandrian scholarship was not all of the same type. The results of Heron were markedly different from those of Euclid or Apollonius or Archimedes, and again there is an abrupt departure from the classical Greek tradition in the extant work of Diophantus. Uncertainty about the life of Diophantus is so great that we do not know definitely in which century he lived. Generally he is assumed to have flourished about AD 250, but dates a century or more earlier or later are sometimes suggested. According to a tradition that is reported in a collection of problems dating from the fifth or sixth century, known as the "Greek Anthology", if it is historically accurate, Diophantus lived to be eighty-four years old.

Diophantus is often called the father of algebra, but we shall see that such a designation is not to be taken literally. His work is not all the type of material forming the basis of modern elementary algebra; nor is it yet similar to the geometric algebra found in Euclid. The chief Diophantine work known to us is the *Arithmetica* , a treatise originally in thirteen books, only the first six of which have survived. It should be recalled that in ancient Greece the word arithmetic meant theory of numbers rather than computation. Often Greek arithmetic had more in common with philosophy than with what we think if as mathematics; hence, the subject had played a large role in Neoplatonism during the Later Alexandrian Age. This had been particularly true of the *Introductio arithmeticae* of Nicomachus of Gerasa, a Neo-Pythagorean who lived not far from Jerusalem about the year 100.

Quite different from the work of Nocomachus, Theon and Boethius was the *Arithmetica* of Diophantus, a treatise characterized by a high degree of mathematical skill and ingenuity. In this respect the book can be compared with the great classics of the earlier Alexandrian Age; yet it has practically nothing in common with these or, in fact, with any traditional Greek mathematics. It represents essentially a new branch and makes use of a different approach. Being divorced from geometric methods, it resembles Babylonian algebra to a large extent. But whereas Babylonian mathematicians had been concerned primarily with the *approximate* solution of *determinate* equations as far as the third degree, the *arithmetica* of Diophantus (such as we have it) is almost entirely devoted to the *exact* solution of equations, both *determinate* and *indeterminate*. Because of the emphasis given in the *Arithmetica* to the solution of indeterminate problems, the name dealing with this topic, sometimes known indeterminate analysis, has since become Diophantine analysis. Since this type of work today is generally part of courses in theory of numbers, rather than elementary algebra, it is not an appropriate basis for regarding Diophantus as the father of algebra. There is another respect, however, in which such a paternity is justified. Algebra now is based almost exclusively on symbolic forms of statement, rather than on the customary written language of ordinary communication in which earlier Greek mathematics, as well as Greek literature, had been expressed. It has been said that three stages in the historical development of algebra can be recognised: (1) the rhetorical or early stage, in which everything is written out fully in words: (2) a syncopated or intermediate stage, in which some abbreviations are adopted; and (3) a symbolic or final stage. Such an arbitrary division of the development of algebra into three stags is, of course, a facile over-simplification; but it can serve effectively as a first approximation to what happened, and within such a framework the *Arithmetica* of Diophantus is to be placed in the second category.

Throughout the six surviving books of the *Arithmetica* there is a systematic use of abbreviations for powers of numbers and for relationships and operations. An unknown number is represented by a symbol resembling the Greek letter ς (perhaps for the last letter of *arithmos*); the square of this appears as Δ^{Υ}, the cube as K^{Υ}, the fourth power, called square-square, as $\Delta^{\Upsilon}\Delta$, the fifth power or square-cube as ΔK^{Υ}, and the sixth power or cube-cube as $K^{\Upsilon}K$. Diophantus was, of course, familiar with the rules of combination equivalent to our laws of exponents, and he had special names for the reciprocals of the first six powers of the unknowns, quantities equivalent to our negative powers. Numerical coefficients were written after the symbols for the powers with which they were associated; addition of terms was understood in the appropriate juxtaposition of the symbols for the terms, and subtraction was represented by a single letter abbreviation placed before the terms to be subtracted. With such a notation Diophantus was in a position to write polynomials in a single unknown almost as concisely as we do today. The expression $2x^4 + 3x^3 - 4x^2 + 5x - 6$, for example, might appear in a form equivalent to SS2 C3 *x*5 M S4 u6, where the English letters S, C, *x*, M and u have been used for 'square', 'cube', the 'unknown', 'mimus' and 'unit', and with our present numerals in place of the Greek alphabetic notation that was used in the days of Diophantus. Greek algebra now no longer was restricted to the first three powers or dimensions, and the identities

$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2 = (ac - bd)^2 + (ad + bc)^2$, which played important roles in medieval algebra and modern trigonometry, appear in the work of Diophantus. The chief difference between the Diophantine syncopation and the modern algebraic notation is in the lack of special symbols for operations and relations, as well as of the exponential notation. These missing elements of notation were largely contributions of the period from the late fifteenth to the early seventeenth centuries in Europe.

If we think primarily of matters of notation, Diophantus has a good claim to be known as the father of algebra, but in terms of motivation and concepts the claim is less appropriate. The *Arithmetica* is not a systematic exposition of the algebraic operations or the algebraic functions or of the solution of algebraic equations. It is instead a collection of some 150 problems, all worked out in terms of specific numerical examples, although perhaps generality of method was intended. There is no postulational development, nor is an effort made to find all possible solutions. In the case of quadratic equations with two positive roots, only the larger is given, and negative roots are not recognised. No clear-cut distinction is made between determinate and indeterminate problems, and even for the latter, for which the number of solutions generally is unlimited, only a single answer is given. Diophantus solved problems involving several unknown numbers by skilfully expressing all unknown quantities, where possible, in terms of only one of them. Two problems from the *Arithmetica* will serve to illustrate the Diophantine approach. In finding two numbers such that their sum is 20 and the sum of their squares is 208, the numbers are not designated as x and y, but as $10 + x$ and $10 - x$ (in terms of our modern notation). Then $(10 + x)^2 + (10 - x)^2 = 208$, hence $x = 2$; so the numbers sought are 8 and 12. Diophantus handled also the analogous problem in which the sum of the two numbers and the sum of the cubes of the numbers are given as 10 and 370 respectively.

In these problems Diophantus is dealing with a determined equation, but he used much the same approach in indeterminate analysis. In one problem it is required to find two numbers such that either when added to the square of the other will yield a perfect square. This is a typical instance of diophantine analysis in which only rational numbers are acceptable as answers. In solving the problem Diophantus did not call the numbers x and y, but rather x

and $2x + 1$. Here the second when added to the square of the first, will yield a perfect square no matter what value one choose for x. Now it is required also that $(2x + 1)^2 + x$ must be a perfect square. Here Diophantus does not point out the infinity of possible answers. He is satisfied to choose a particular case of a perfect square, in this instance the number $(2x – 2)^2$, such that when equated to $(2x + 1)^2 + x$ an equation that is linear in x results. Here the result is $x = \frac{3}{13}$, so that the other number, $2x + 1$, is $\frac{19}{13}$. One could, of course, have used $(2x – 3)^2$ or $(2x – 4)^2$, or expressions of similar form, instead of $(2x – 2)^2$, to arrive at other pairs of numbers having the desired property. Here we see an approach that comes close to a 'method' in Diophantus' work: When two conditions are to be satisfied by two numbers, the two numbers are so chosen that one of the two conditions is satisfied; and then one turns to the problem of satisfying the second condition. That is, instead of handling *simultaneous* equations on two unknowns, Diophantus operates with *successive* conditions so that only a single unknown number appears in the work.

Among the indeterminate problems in the *Arithmetica* are some involving equations such as $x^2 = 1 + 30y^2$ and $x^2 = 1 + 26y^2$, which are instances of the so-called "Pell-equation" $x^2 = 1 + py^2$; again a single answer is though to suffice. In a sense it is not fair to criticise Diophantus for being satisfied with a single answer, for he was solving problems, not equations. In a sense the *Arithmetica* is not an algebra textbook, but a problem collection in the application of algebra. In this respect Diophantus is like the Babylonian algebraists; and his work sometimes is regarded as "the finest flowering of Babylonian algebra." (Swift 1956). To some extent such a characterisation is unfair to Diophantus, for his numbers are entirely abstract and do not refer to measures of grain or dimensions of fields or monetary units, as was the case in Egyptian and Mesopotamian algebra. Moreover, he is interested only in *exact* rational solutions, whereas the Babylonians were computationally inclined and were willing to accept approximations to irrational solutions to equations. Hence, cubic equations seldom enter the work of Diophantus, whereas among the Babylonians attention had been given to the reduction of cubics to the standard form $n^3 + n^2 = a$ in order to solve approximately through interpolation in a table of values of $n^3 + n^2$.

We do not know how many of the problem in the *Arithmetica* were original or whether Diophantus had borrowed from other similar collections. Possibly some of the problems or methods are traceable back to Babylonian sources, for puzzles and exercises have a way of reappearing generation after generation. To us today the *Arithmetica* of Diophantus looks strikingly original, but possibly this impression results from the loss of rival problem collections. Our view of Greek mathematics is derived from a relatively small number of surviving works, and conclusions derived from these necessarily are precarious. Indications that Diophantus may have been less isolated a figure than has been supposed are found in a collection of problems from about the early second century of our era (hence presumably antedating the *Arithmetica*) in which some Diophantine symbols appear.

Nevertheless, Diophantus has had a greater influence on modern number theory than any other nongeometric Greek algebraist. In particular, Fermat was led to his celebrated “great” or “last” theorem when he sought to generalise a problem that he had read in the *Arithmetica* of Diophantus (II:8): to divide a given square into two squares.

[Extracts from Carl B. Boyer *A History of Mathematics* (2nd Edition) p 178 - 183]

Appendix 3

François Viète (Franciscus Vieta) (1540 - 1603)

French mathematician noted for his *In artum analyticam isagoge* (1591; Introduction to the Analytical Arts), one of the earliest Western works on algebra. In it he denoted unknowns by vowels and known quantities by consonants, and also introduced an improved notation for squares, cubes and other powers. With his new algebraic techniques Viète succeeded in solving a number of problems classical authors had found unyielding to geometrical attacks. He developed new methods of solving of solving equations in his *De aequantionum recognitione et emendatione* (1615; On the Recognition and Emendation of Equations). He was the first, in his *Canon mathematicus seu ad triangula* (1579: The Mathematical Canon Applied to Triangles), to tackle the problem of solving plane and spherical triangles with the help of the six main trigonometrical functions.

From: *The Penguin Dictionary of Mathematics*, 1989, page 339

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The Frenchman François Viète was the first in his '*logistica speciosa'* to introduce letters as general (positive) coefficients and to put some other finishing touches on symbolism, which was finally up to date by the time of Isaac Newton (1642 - 1727). But Viète's most significant contributions were contained in *De aequantionum recognitione et emendatione*, published posthumously in 1615. In this work he:

1. Gave transformations for increasing or multiplying the roots of an equation by a constant.
2. Indicated awareness of relations between the roots and coefficients of polynomial equations.
3. Stated a transformation that rids a polynomial of its next-to-highest-degree term.

Viète's inability to accept negative numbers (not to mention imaginary numbers) prevented him from attaining the generality he sought (and partly comprehended) in giving, for example, relations between the roots and the coefficients of a polynomial equation.

An example of the notation used by Viète

I QC - 15 QQ + 85C - 225Q + 274N aequatur 120

In our notation this means:

$^{-}15x^4 + 85x^3 - 225x^2 + 274x = 120$

From: NCTM Historical Topics for the Mathematics Classroom, 1989, page 245/263

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In 1560, François Viète, a student of law in France, served Henri IV of Navarra in the war against Spain by decoding intercepted letters which were written in code. He went on to develop interest in astronomy and plane and spherical trigonometry, and published a book in which he introduced the use of letters not only for unknowns but also to represent known quantities

From: Open University, PM 753C, *Preparing to teach Equations*, page 32.

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Without doubt it was in algebra that Viète made his most estimable contributions, for it was here that he came close to modern views. Mathematics is a form of reasoning, and not a bag of tricks, such as Diophantus had possessed; yet algebra, during the Arabic and early modern periods, had not gone far from freeing itself from the treatment of special cases. There could be little advance in algebraic theory so long as the chief occupation was with finding "the thing" in an equation with specific numerical coefficients. Symbols and abbreviations for an unknown, and for powers of the unknown, as well as for operations and for the relationship of equality had to be developed

Here Viète introduced a convention as simple as it was fruitful. He used a vowel to represent the quantity in algebra that was assumed to be unknown or underdetermined and a consonant to represent a magnitude or number assumed to be known or given. Here we find for the first time in algebra a clear-cut distinction between the important concept of a parameter and the idea of an unknown quantity.

Had Viète adapted other symbolisms extant in his day, he might have written *all* quadratic equations in the single form $BA^2 + CA + D = 0$, where A is the unknown and B, C and D are parameters; but unfortunately he was modern only in some ways and ancient and medieval in others. His algebra is fundamentally syncopated rather than symbolic, for although he wisely adopted the Germanic symbols for addition and subtraction and, still more wisely, used differing symbols for parameters and unknowns, the remainder of his algebra consisted of words and abbreviations. The third power of the unknown was not A^3, or even AAA, but *A cubus*, and the second power was *A quadratus*. Multiplication was signified by the Latin word *in*, division was indicated by the fraction line, and for equality Viète used an abbreviation for the Latin *aequalis*. It is not given for one man to make the whole of a given change; it must come in steps.

One of the steps beyond the work of Viète was taken by Harriot when he revived the idea Stidel had had of writing the cube of the unknown as AAA. This notation was used systematically by Harriot in his posthumous book entitles *Artis analyticae praxis* and printed in 1631. Its title had been suggested by the earlier work of Viète, who had disliked the Arabic name algebra.

In view of the type of reasoning so frequently used in algebra, Viète called the subject "the analytic art". Moreover, he had a clear awareness of the broad scope of the subject, realising that the unknown quantity need not be either a number or a geometric line. Algebra reasons about “types” or species, hence Viète contrasted *logista speciosa* with *logista numerosa*. His algebra was presented in the *Isagoge* (or *Introduction*), printed in 1591, but his several other algebraic works did not appear until many years after his death. In all of these he maintained a principle of homogeneity in equations, so that in an equation such as $x^3 + 3ax = b$ the *a* is designated as *planum* and the *b* as *solidum*. This suggest a certain inflexibility, which Descartes removed a generation later; but homogeneity also has certain advantages, as Viète undoubtedly saw.

The algebra of Viète is noteworthy for the generality of its expressions, but there are also other novel aspects. For one thing, Viète suggested a new approach to the solution of the cubic. Having reduced it to the standard equivalent of $x^3 + 3ax = b$, he introduced a new unknown quantity *y* that was related to *x* through the equation in y^3, for which the solution is readily obtained. Moreover, Viète was aware of some the relations between roots and coefficients of an equation, although here he was hampered by his failure to allow the coefficients and roots to be negative.

Extracts from: *A History of Mathematics* (2nd Edition), C.B. Boyer, page 303-305