Convexity Adjustment – Effect of Stochastic Rate on Forward of Driftless Asset

Ranjit Kumar[1*]

[1] *Working as Quantitative Analyst in Investment Bank*

Mumbai, India.

Abstract

We provide a straightforward calculation for the convexity adjustment of the forward on driftless asset using the Hull-White model for interest rate in the T-forward measure. Calculation in bond numeraire is also provided. In previous article we showed that the convexity adjustment manifests when an interest rate is paid out in the ”wrong” currency [7]. The convexity adjustment also results from higher order derivative in stochastic calculations due to the use of Ito's calculus. In general, the convexity adjustments depends on the variance (implied volatility), correlation and time to maturity of the underlying process.

Keywords: Stochastic Process, Change of Measure, Numeraire

* Corresponding Author's
E-Mail: du.ranjit@gmail.com
Tel: 91-9650289875

I. INTRODUCTION

We provide a simple derivation for the convexity adjustment of the forward on driftless asset using the Hull-White model for interest rate in both the money market (risk-neutral measure) and bond (T-forward measure) numeraire [1–8]. We demonstrate that the convexity adjustment depends on the implied volatility of the assets and correlation ρ between the interest rate and asset price as well time to maturity of the trade. The convexity adjustment is zero for deterministic rates. Furthermore, the convexity adjustment is zero when the asset is not driftless with mean rate of return r. Note that in all cases, we assume that the asset return follow geometric Brownian motion. The next article will provide the calculation for asset return following the arithmetic Brownian motion (ABM). The outline of the article is as follows: Fundamental of stochastic calculus is discussed in Section II. This section also discusses the fundamental theorem of asset pricing and change of numeraire. Forwards are discussed in Section III. Convex function are briefly discussed in Section IV. The calulation in bond numeraire and money market numeraire of forward on driftless asset is provided in Section V. We summarise the results in Section VI.

II. FUNDAMENTAL OF STOCHASTIC CALCULUS

Money Market

Let risk free rate $r(t)$ is stochastic. Then the money account can be written as

$$\begin{aligned} B(t) &= B(0)e^{\int_0^t r(s)\ ds} \\ dB(t) &= r(t)B(t)\ dt \end{aligned}$$

Note that there is no $W(t)$ term in the $dB(t)$ equation.

Martingale

Consider a random variable $S(t)$. The random variable $S(t)$ is martingale if

$$S(t) = E^{\mathbb{Q}}\left[S(T)|\mathcal{F}_t\right]$$

In other words, a martingale is a sequence of random variables in which the conditional expected value (conditional on the current value) of the next variable, is the current value.

Numeraire

A numeraire $N(t)$ is any tradeable asset, such that $N(t) > 0$. The relative price of an asset $S(t)$ is defined by

$$S^N(t) = \frac{S(t)}{N(t)}$$

In other words, numeraire is a unit in which the price of an asset is expressed. Note that we will take only non-dividend paying asset as numeriare. Some of the most used examples of numeraire are

- **Domestic Money Market Account:** The associated risk neutral measure is denoted by $\mathbb{Q}^d$
- **Foreign Money Market Account:** The associated risk neutral measure is denoted by $\mathbb{Q}^f$
- **Bond of Maturity T:** The associated risk neutral measure is denoted by T.
- **Non-Dividend Paying Stock Price:** The associated risk neutral measure is denoted by $\mathbb{S}$.

A. Fundamental Theorem of Asset Pricing

First Fundamental Theorem of Arbitrage Pricing

The market is arbitrage free if and only if there exist an equivalent martingale measure $\mathbb{N}$.

- The arbitrage free market means the existence of a numeraire $N(t)$ and an equivalent measure $\mathbb{N}$.

An important consequence of this theorem is the arbitrage pricing law, i.e.,

$$\frac{S(t)}{N(t)} = E^{\mathbb{N}}\left[\left.\frac{S(t)}{N(T)}\right|\mathcal{F}_t\right]$$

In other words, the relative price of the asset $\frac{S(t)}{N(t)}$ is a martingale. This relation is also true for the derivative $V(t)$, i.e.,

$$\frac{V(t)}{N(t)} = E^{\mathbb{N}}\left[\left.\frac{V(T)}{N(T)}\right|\mathcal{F}_t\right]$$

If the numeraire is money market then

$$V(t) \;=\; E^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}V(T)\middle|\mathcal{F}(t)\right] = E^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\;du}\;V(T)\middle|\mathcal{F}(t)\right]$$

When the derivative contract is bond, then

$$V(t) = P(t,T) \qquad \text{and} \qquad V(T) = P(T,T) = 1$$

and hence

$$P(t,T) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\;du}\middle|\mathcal{F}(t)\right]$$

Second Fundamental Theorem of Arbitrage Pricing

An arbitrage free market is complete if and only if the equivalent martingale measure $\mathbb{N}$ is unique.

- In a complete arbitrage free market, to a given numeraire $N(t)$ corresponds to one and only one martingale measure $\mathbb{N}$.

Thus in a complete market, for each numeraire $N(t)$, there exist a unique equivalent martingale measure $\mathbb{N}$. If we chose another numeraire $M(t)$, then $\frac{S(t)}{M(t)}$ is no longer a martingale under $\mathbb{N}$. But the market completeness assures that there exist another equivalent martingale measure $\mathbb{M}$, such that $\frac{S(t)}{M(t)}$ is a martingale under $\mathbb{M}$. Therefore

$$\frac{S(t)}{M(t)} = E^{\mathbb{M}}\left[\frac{S(t)}{M(T)}\middle|\mathcal{F}_t\right]$$

B. Numarare and Change of Measure

With the help of Radon-Nikodym derivative we can pass from measure $\mathbb{N}$ to measure $\mathbb{M}$. The change of measure in terms of numeraire $N(t)$ and $M(t)$ can be expressed as

$$L(t) = \frac{d\mathbb{M}}{d\mathbb{N}} = \frac{\frac{N(t)}{N(T)}}{\frac{M(t)}{M(T)}}$$

Proof: We know that

$$S(t) = E^{\mathbb{N}}\left[S(T)\frac{N(t)}{N(T)}\middle|\mathcal{F}_t\right] = E^{\mathbb{M}}\left[S(T)\frac{M(t)}{M(T)}\middle|\mathcal{F}_t\right]$$

Now change the measure on right hand side, we obtain

$$E^{\mathbb{N}}\left[S(T)\frac{N(t)}{N(T)}\bigg|\mathcal{F}_t\right] = E^{\mathbb{N}}\left[S(T)\frac{M(t)}{M(T)}\ \frac{d\mathbb{M}}{d\mathbb{N}}\bigg|\mathcal{F}_t\right]$$

Equating the terms inside the expectation, we obtain

$$\frac{d\mathbb{M}}{d\mathbb{N}} = \frac{\frac{N(t)}{N(T)}}{\frac{M(t)}{M(T)}}$$

Thus with the help of Radon-Nikodỳm derivative we can go back and forth between the two equivalent martingale measure.

1. Forward and Bond Measure

In forward measure, the price of derivative $V(t)$ at time t is given by

$$\frac{V(t)}{P(t,T)} = E^{\mathbb{T}}\left[\frac{V(T)}{P(T,T)}\bigg|\mathcal{F}_t\right] \tag{1}$$

From here we can write

$$V(t) = P(t,T)\ E^{\mathbb{T}}\left[\frac{V(T)}{P(T,T)}\bigg|\mathcal{F}_t\right]$$

and in bond measure, the price of asset is

$$\frac{V(t)}{B(t)} = E^{\mathbb{B}}\left[\frac{V(T)}{B(T)}\right]$$

which can be written as

$$V(t) = B(t)\ E^{\mathbb{B}}\left[\frac{V(T)}{B(T)}\right]$$

Comparing the above two equations, we obtain

$$P(t,T)\ E^{\mathbb{T}}\left[\frac{V(T)}{P(T,T)}\right] = B(t)\ E^{\mathbb{B}}\left[\frac{V(T)}{B(T)}\right]$$

Therefore,

$$E^{\mathbb{T}}\left[V(T)\right]\ =\ E^{\mathbb{B}}\left[V(T)\cdot\frac{B(t)}{B(T)}\cdot\frac{P(T,T)}{P(t,T)}\right] = E^{\mathbb{B}}\left[V(T)\cdot\frac{dQ^{\mathbb{T}}}{dQ^{\mathbb{B}}}\right]$$

Hence

$$\frac{dQ^{\mathbb{T}}}{dQ^{\mathbb{B}}} = \frac{\frac{B(t)}{B(T)}}{\frac{P(t,T)}{P(T,T)}}$$

2. *T-Forward Measure*

In T-forward measure, $P(t,T)$ is used as numeraire. The equivalent martingale measure associated with using $P(t,T)$ as numeraire is also known as T-forward measure. In T-forward measure, we use $P(t,T)$ as numeraire

$$\frac{V(t)}{P(t,T)} = E^{\mathbb{T}}\left[\left.\frac{V(T)}{P(T,T)}\right|\mathcal{F}(t)\right]$$

Sine $P(T,T)=1$, we get

$$V(t) \; = \; P(t,T)E^{\mathbb{T}}\left[V(T)|\,\mathcal{F}(t)\right] = E^{\mathbb{Q}}\left[\left.e^{-\int_t^T r(u)\ du}\right|\mathcal{F}(t)\right]E^{\mathbb{T}}\left[V(T)|\,\mathcal{F}(t)\right]$$

Also

$$V(t) \; = \; E^{\mathbb{Q}}\left[\left.\frac{B(t)}{B(T)}V(T)\right|\mathcal{F}(t)\right] = E^{\mathbb{T}}\left[P(t,T)V(T)|\,\mathcal{F}(t)\right]$$

Therefore,

$$E^{\mathbb{Q}}\left[\left.\frac{B(t)}{B(T)}V(T)\right|\mathcal{F}(t)\right] = E^{\mathbb{T}}\left[\left.\frac{P(t,T)}{P(T,T)}V(T)\right|\mathcal{F}(t)\right]$$

The Radon-Nikodym derivative can be obtained as follows:

$$\begin{aligned} E^{\mathbb{T}}\left[\left.\frac{P(t,T)}{P(T,T)}V(T)\right|\mathcal{F}(t)\right] &= E^{\mathbb{Q}}\left[\left.\frac{P(t,T)}{P(T,T)}V(T)\cdot\frac{dT}{dQ}\right|\mathcal{F}(t)\right] \\ &= E^{\mathbb{Q}}\left[\left.\frac{B(t)}{B(T)}V(T)\right|\mathcal{F}(t)\right] \end{aligned}$$

where

$$E^{\mathbb{T}}[X] = E^{\mathbb{Q}}\left[\left.X\frac{dT}{dX}\right|\mathcal{F}(t)\right] \tag{2}$$

Comparing these two equations, we obtain

$$\frac{dT}{dQ} = \frac{\frac{B(t)}{B(T)}}{\frac{P(t,T)}{P(T,T)}} = \frac{e^{-\int_t^T r(u)\ du}}{P(t,T)}$$

3. *Unnatural Rates*

Convexity adjustment manifests when an interest rate is paid out at the "wrong" time or in the "wrong" currency [6, 9]. Consider the scenario where a zero coupon bond with a time to maturity of T is the numeraire and we have an unnatural Libor payout X_S with time to maturity S that is to be evaluated as an expectation under an unnatural terminal forward measure $\mathbb{T}$. There is clearly a timing mismatch between the payoff with maturity S and the terminal forward process $\mathbb{T}$ with zero coupon numberaire with maturity T with $T < S$. We know that

$$V(t) = P(t,T)E^{\mathbb{T}}\left[V(T)|\,\mathcal{F}(t)\right]$$

Under the unnatural measure, we wrongly estimate the price by utilising this equation.

$$V(t) = P(t,T)E^{\mathbb{T}}\left[\overline{V}(S)\middle|\,\mathcal{F}(t)\right], \qquad t < T < S \tag{3}$$

Note that the convexity adjustments for unnatural rates are due to timing lags in the rate process rather than the payment date. We denote unnatural rate or payoff by $\overline{V}(T)$. Whilst payment is made at time T the underlying rate has a timing adjustment such as a fixing lag. It is no longer a martingale with respect to the measure being used. Now using Eq. (2), we can change the measure in the expectation within (3) to a martingale measure as follows,

$$\underbrace{E^{\mathbb{T}}\left[\overline{V}(T)\middle|\,\mathcal{F}(t)\right]}_{\text{Not a martingale}} = \underbrace{E^{\mathbb{Q}}\left[V(T)\frac{dT}{dQ}\middle|\,\mathcal{F}(t)\right]}_{\text{Martingale}}$$

Let us choose $P(t,T)$ be the numeraire of the unnatural measure $\mathbb{T}$ and $B(t)$ for the natural measure Q evaluated at time t. Since

$$\frac{dT}{dQ} = \frac{\frac{B(t)}{B(T)}}{\frac{P(t,T)}{P(T,T)}} = \frac{B(t)P(T,T)}{B(T)P(t,T)}$$

Therefore

$$\begin{aligned} E^{\mathbb{T}}\left[\overline{V}(T)\middle|\,\mathcal{F}(t)\right] &= E^{\mathbb{Q}}\left[V(T)\frac{dT}{dQ}\middle|\,\mathcal{F}(t)\right] = E^{\mathbb{Q}}\left[V(T)\frac{B(t)P(T,T)}{B(T)P(t,T)}\middle|\,\mathcal{F}(t)\right] \\ &= E^{\mathbb{Q}}\left[V(T)R(T)|\,\mathcal{F}(t)\right] = E^{\mathbb{Q}}\left[V(T)\frac{G(T)}{G(t)}\middle|\,\mathcal{F}(t)\right] \end{aligned}$$

where

$$R(T) = \frac{G(T)}{G(t)}$$

and

$$G(t) = \frac{P(t,T)}{B(t)}$$

We can further simplify above equation as

$$\begin{aligned} E^{\mathbb{T}}\left[\overline{V}(T)\middle|\,\mathcal{F}(t)\right] &= E^{\mathbb{Q}}\left[V(T)R(T)|\,\mathcal{F}(t)\right] \\ &= E^{\mathbb{Q}}\left[V(T)-V(T)+V(T)R(T)|\,\mathcal{F}(t)\right] \\ &= E^{\mathbb{Q}}\left[V(T)|\,\mathcal{F}(t)\right]+E^{\mathbb{Q}}\left[V(T)\left(R(T)-1\right)|\,\mathcal{F}(t)\right] \\ &= E^{\mathbb{Q}}\left[V(T)|\,\mathcal{F}(t)\right]+E^{\mathbb{Q}}\left[V(T)\left(\frac{G(T)}{G(t)}-1\right)\middle|\,\mathcal{F}(t)\right] \end{aligned}$$

For a $\mathbb{Q}$-martingale process, the convexity adjustment is therefore given by

$$CA = E^{\mathbb{Q}}\left[V(T)\left(\frac{G(T)}{G(t)}-1\right)\middle|\,\mathcal{F}(t)\right]$$

We need joint density function of $V(T)$ and $R(T)$ to estimate the convexity adjustment. Please see Ref [6, 9] for detailed discussion on this topic.

III. FORWARD

A forward contract is a non-standardized contract between two parties and they are traded in the OTC market. In forward contract one party takes the long position and agrees to buy the underlying asset at a specified price on the specified date, while the other party takes the short position and agrees to sell the asset on the same date at the same price.

A. Forward Price

It is an agreement to pay a specified delivery price K at time T for an asset whose price at time t is $S(t)$. We consider the following strategy: buy the stock S and sell K zero-coupon bonds with maturity T. The value of portfolio at time t is given by

$$\Pi_t = S(t) - KP(t,T)$$

The value of portfolio at time T is

$$\Pi_T = S(t) - KP(T,T) = S(t) - K$$

The forward price $F(t,T)$ at time t is the value of K that makes the contract price zero. Hence

$$F(t,T) = \frac{S(t)}{P(t,T)}$$

If r is constant, then

$$F(t,T) = S(t)\ e^{r(T-t)}$$

The risk neutral price of forward contract is given by

$$\frac{V(t,T)}{B(t)} = E^Q\left[\frac{V(T,T)}{B(T)}\right] = E^Q\left[\frac{(S(t)-K)}{B(T)}\middle|\mathcal{F}(t)\right]$$

From here we get

$$\begin{aligned} V(t,T) &= E^Q\left[B(t)\frac{V(T,T)}{B(T)}\middle|\mathcal{F}(t)\right] \\ &= E^Q\left[e^{-\int_t^T r(u)\ du}(S(t)-K)\middle|\mathcal{F}(t)\right] \end{aligned}$$

The forward price $V(t,T)$ at time t is the value of K that makes the contract price zero.

$$E^Q\left[e^{-\int_t^T r(u)\ du}\,(S(T)-K)\middle|\mathcal{F}(t)\right] = 0$$

From here we get

$$K = \frac{E^Q\left[e^{-\int_t^T r(u)\ du}S(T)\middle|\mathcal{F}(t)\right]}{E^Q\left[e^{-\int_t^T r(u)\ du}\middle|\mathcal{F}(t)\right]}$$

In T-forward measure, this equation may be written as

$$K = \frac{E^T\left[S(T)|\,\mathcal{F}(t)\right]\cdot P(t,T)}{P(t,T)} = E^T\left[S(T)|\,\mathcal{F}(t)\right]$$

1. *Impact of Stochastic Rates On Forward of Equity-Like Asset*

In risk-neutral world, the stochastic differential equation for equity may be written as

$$dS(t) = r(t)S(t)\ dt + \sigma_s\ dW^s(t)$$

where σ_s is assumed to be constant. Let us define a new stochastic process

$$Z = \ln S$$

Then using Itô's lemma, one can show that

$$dZ(t) = \left(r(t) - \frac{\sigma_s^2}{2}\right) dt + \sigma_s\ dW^s(t)$$

Integrating this equation in the limit t to T, we obtain

$$S(T) = S(t)e^{\int_t^T r(u)\ du - \frac{\sigma_s^2}{2}(T-t) + \int_t^T \sigma_s\ dW^s(u)}$$

Therefore, the forward price for the equity like asset may be written as

$$\begin{aligned}
F(t,T) &= \frac{E^Q\left[e^{-\int_t^T r(u)\ du} S(T) \middle| \mathcal{F}(t)\right]}{P(t,T)} \\
&= \frac{E^Q\left[e^{-\int_t^T r(u)\ du}\ S(t)\ e^{\int_t^T r(u)\ du - \frac{\sigma_s^2}{2}(T-t) + \int_t^T \sigma_s\ dW^s(u)} \middle| \mathcal{F}(t)\right]}{P(t,T)} \\
&= \frac{E^Q\left[S(t) e^{-\frac{\sigma_s^2}{2}(T-t) + \int_t^T \sigma_s\ dW^s(u)} \middle| \mathcal{F}(t)\right]}{P(t,T)} = \frac{S(t)}{P(t,T)}
\end{aligned}$$

where

$$E^Q\left[e^{\int_t^T \sigma_s\ dW^s(u)} \middle| \mathcal{F}(t)\right] = e^{\frac{\sigma_s^2}{2}\ (T-t)}$$

Hence forward price remains unchanged for stochastic interest rates. Similarly, in the T-forward measure, we have

$$E^T\left[S(T)|\mathcal{F}(t)\right] = E^T\left[\frac{S(T)}{P(T,T)}\right] = \frac{S(t)}{P(t,T)}$$

Thus the answer remains unchanged.

2. Forward Curve From an Asset Price

Let the asset follow the geometric Brownian motion

$$dS(t) = \mu\ S(t)\ dt + \sigma\ S(t)\ dW_t$$

The forward price of asset at time t is given by

$$F(t,T) = E[S(t)|\mathcal{F}_t]$$

Show that forward curve follows

$$\frac{dF(t,T)}{F(t,T)} = \sigma\ dW_t$$

Proof: Since

$$S(t) = S(t)\ e^{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma(W_T - W_t)}$$

Then

$$E[S(t)|S(t)=s]\ =\ e^{\ln S(t) + \left(\mu - \frac{\sigma^2}{2}\right)\tau + \frac{\sigma^2}{2}(T-t)} = S(t)\ e^{\mu(T-t)}$$

Hence

$$F(t,T) = E[S(t)|\mathcal{F}_t] = S(t)\ e^{\mu(T-t)}$$

Furthermore,

$$\begin{aligned} dF(t,T) &= \frac{\partial F}{\partial t}\ dt + \frac{\partial F}{\partial S(t)}\ dS(t) + \frac{1}{2}\frac{\partial^2 F}{\partial S(t)^2}\ dS(t)^2 + \cdots \\ &= -\mu S(t)\ e^{\mu(T-t)}\ dt + e^{\mu(T-t)}\ dS(t) \\ &= \sigma\ S(t)\ e^{\mu(T-t)}\ dW_t = \sigma\ F(t,T)\ dW_t \end{aligned}$$

Hence

$$\frac{dF(t,T)}{F(t,T)} = \sigma\ dW_t$$

The forward on asset follows driftless geometric Brownian motion.

IV. CONVEXITY

In this section we will briefly discuss the convex function. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

where $\lambda \in [0,1]$.

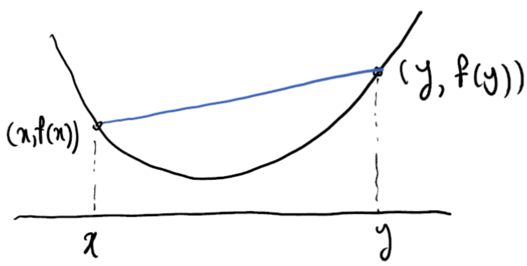

FIG. 1: Example of a Convex Function. Geometrically, the line segment between any two points on the graph of the function lies above the graph between the two points.

A. Examples of Convex Function

- e^{ax}
- $-\ln(x)$

Note that if $f(x)$ has a second derivative in $[a, b]$, then a necessary and sufficient condition for it to be convex on that interval is that the second derivative $f^{''}(x) \geq 0$ for all x in $[a, b]$. As an example let us consider Weiner process $W(t)$ and a function $f = f(W(t))$. Then using the Itô-Doeblin formula [2], we get

$$df(W(t)) = \frac{\partial f(W(t))}{\partial W(t)}\ dW(t) + \frac{1}{2}\frac{\partial^2 f(W(t))}{\partial W(t)^2}(dW(t))^2 + \text{Small Error}$$

Note that the second order term $\frac{1}{2}f^{''}(W(t)(dW(t))^2$ capture the curvature of the function. Let

$$f(W(t)) = W^2(t)$$

Then

$$df(W(t)) = 2W(t)\ dW(t) + dt + \mathcal{O}\left(t^{3/2}\right)$$

B. Convexity of Stock's Price – Adjustment Due to Itô's Lemma

Let the stochastic differential equation for the asset is assumed to be of the form

$$dS(t)\ =\ r(t)\ dt + S(t)\ \sigma_s\ dW^s(t)$$

Let us define

$$f(S(t)) = \ln(S(t))$$

Again using the Itô-Doeblin formula [2], we get

$$df(S(t)) = \frac{\partial f(S(t))}{\partial S(t)}\ dS(t) + \frac{1}{2}\frac{\partial^2 f(S(t))}{\partial S(t)^2}(dS(t))^2 + \text{Small Error}$$

From here we get

$$df(S(t)) = r\ dt + \sigma_s\ dW^s(t) - \frac{1}{2}\sigma_s^2\ dt + \mathcal{O}\left(t^{3/2}\right)$$

Integrating this equation, we obtain

$$S(t) = S(0)\ e^{\left(r - \frac{\sigma_s^2}{2}\right)t + \sigma_s\ W^s(t)}$$

Since the Brownian motion $W^s(t)$ is martingale, the function

$$S(0)\ e^{\sigma_s W^s(t)}$$

is not a martingale but

$$S(0)\ e^{\sigma_s W^s(t) - \frac{1}{2}\sigma_s^2 t}$$

is a martingale and hence the mean rate of return of asset is not equal to $r - \frac{\sigma^2}{2}$, it is equal to r. Note that the convexity of the function $e^{\sigma x}$ imparts an upward drift to $S(0)\ e^{\sigma_s W^s(t)}$. The correction term is

$$S(0)\ e^{-\frac{\sigma_s^2}{2} t}$$

Note that this equation can also be written as

$$S(0)\ e^{-\frac{\sigma_s \times \sigma_s \times \rho^{s,s}}{2}\ t}$$

where $\rho^{s,s} = 1$, i.e., the correlation of stock return with itself. Thus the convexity adjustment depends on the product of implied volatility of the underlying, the correlation between the underlying and the time to maturity t. We will see this relation later in the Section V. If $r = 0$, then

$$dS(t) \ = \ S(t)\ \sigma_s\ dW^s(t)$$

Then

$$S(t) = S(0)\ e^{\sigma_s\ W^s(t) - \frac{\sigma_s^2}{2} t}$$

is a martingale. Let us also consider the case of arithmetic Brownian motion

$$dS(t) = \mu\ dt + \sigma_s\ dW^s(t)$$

Note that there is no convexity adjustment in this case because we do not need a transformation like in the previous case.

V. IMPACT OF STOCHASTIC RATES ON FORWARD OF A DRIFTLESS ASSET

A. Calculation In Risk Neutral Measure

Consider the case of forward on a future index. The stochastic differential equation for the asset and the rate is assumed to be of the form

$$\begin{aligned} dS(t) &= S(t)\ \sigma_s\ dW^s(t) \\ dr_d(t) &= -\theta_d \left[r_d(t) - \overline{r}_d(t) \right] dt + \sigma_d\ dW^d(t) \\ \left\langle W^s(t),\ W^d(t) \right\rangle &= \rho^{s,d}\ t \end{aligned}$$

Integrating the first equation, we obtain

$$S(T) = S(t)\ e^{-\frac{\sigma_s^2}{2}\ (T-t)+\sigma_s[W^s(T)-W^s(t)]}$$

To integrate the second equation, we first multiply it by the integrating factor $e^{\theta_d t}$, i.e.,

$$\begin{aligned}
d\left[e^{\theta_d t}\ r_d(t)\right] &= e^{\theta_d t}\ r_d(t)\ dt + e^{\theta_d t}\ dr_d(t) \\
&= e^{\theta_d t}\ r_d(t)\ dt + e^{\theta_d t}\left[-\theta_d\left[r_d(t) - \overline{r}_d(t)\right]dt + \sigma_d\ dW^d(t)\right] \\
&= \theta_d\ \overline{r}_d(t)\ e^{\theta_d t}\ dt + \sigma_d\ e^{\theta_d t}\ dW^d(t)
\end{aligned}$$

Integrate from t to s, we obtain

$$\int_t^s d\left[e^{\theta_d t}\ r_d(t)\right] = \int_t^s \theta_d\ \overline{r}_d(u)\ e^{\theta_d u}\ du + \int_t^s \sigma_d\ e^{\theta_d u}\ dW^d(u)$$

This in turn yields

$$r_d(s) = r_d(t)e^{-\theta_d(s-t)} + \int_t^s \theta_d\ \overline{r}_d(u)\ e^{-\theta_d(s-u)}\ du + \sigma_d \int_t^s e^{-\theta_d(s-u)}\ dW^d(u)$$

Again integrate from t to T with respect to s and change the order of integration

$$\begin{aligned}
\int_t^T r_d(s)\ ds &= \int_t^T r_d(t)e^{-\theta_d(s-t)}\ ds + \int_t^T \left\{\int_t^s \theta_d\ \overline{r}_d(u)\ e^{-\theta_d(s-u)}\ du\right\} ds + \\
&\quad \sigma_d \int_t^T \left\{\int_t^s e^{-\theta_d(s-u)}\ dW^d(u)\right\} ds \\
&= r_d(t)\ b_d(t,T) + a_d(t,T) + \sigma_d \int_t^T b_d(s,T)\ dW^d(s)
\end{aligned}$$

where

$$\begin{aligned}
\int_t^T r_d(t)e^{-\theta_d(s-t)}\ ds &= -\frac{r_d(t)}{\theta_d}\ e^{-\theta_d(s-t)}\Big|_t^T \\
&= \frac{r_d(t)}{\theta_d}\left[1 - e^{-\theta_d(T-t)}\right] = r_d(t)\ b_d(t,T)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_t^T \left\{\int_t^s \theta_d\ \overline{r}_d(u)\ e^{-\theta_d(s-u)}\ du\right\} ds &= \int_t^T \left\{\int_t^u \theta_d\ \overline{r}_d(u)\ e^{-\theta_d(s-u)}\ ds\right\} du \\
&= -\int_t^T \theta_d\ \overline{r}_d(u)\ b_d(u,t)\ du
\end{aligned}$$

and

$$\begin{aligned}
\sigma_d \int_t^T \left\{\int_t^s e^{-\theta_d(s-u)}\ dW^d(u)\right\} ds &= \sigma_d \int_t^T \left\{\int_t^u e^{-\theta_d(s-u)}\ ds\right\} dW^d(u) \\
&= -\sigma_d \int_t^T b_d(u,t)\ dW^d(u)
\end{aligned}$$

Finally, the expression for discounting factor may be written as

$$e^{-\int_t^T r_d(s)\ ds} = e^{-r_d(t)\ b_d(t,T) - a_d(t,T) + \sigma_d \int_t^T b_d(s,T)\ dW^d(s)}$$

Then

$$\begin{aligned} E^Q\left[e^{-\int_t^T r_d(s)\ ds}\right] &= E^Q\left[e^{-r_d(t)\ b_d(t,T) - a_d(t,T) + \sigma_d \int_t^T b_d(s,T)\ dW^d(s)}\right] \\ &= e^{-r_d(t)\ b_d(t,T) - a_d(t,T) + \frac{\sigma_d^2}{2} \int_t^T b_d^2(s,T)\ ds} \end{aligned}$$

Since

$$\begin{aligned} P(t,T) &= E^Q\left[e^{-\int_t^T r_d(s)\ ds}\right] \\ &= e^{-r_d(t)\ b_d(t,T) - a_d(t,T) + \frac{\sigma_d^2}{2} \int_t^T b_d^2(s,T)\ ds} \end{aligned}$$

It can be written as

$$P(t,T)\ e^{-\frac{\sigma_d^2}{2} \int_t^T b_d^2(s,T)\ ds} = e^{-r_d(t)\ b_d(t,T) - a_d(t,T)}$$

Hence

$$\begin{aligned} P(t,T)\ e^{-\frac{\sigma_d^2}{2} \int_t^T b_d^2(s,T)\ ds - \sigma_d \int_t^T b_d(s,T)\ dW^d(s)} &= e^{-r_d(t)\ b_d(t,T) - a_d(t,T) - \sigma_d \int_t^T b_d(s,T)\ dW^d(s)} \\ &= e^{-\int_t^T r_d(s)\ ds} \end{aligned} \tag{4}$$

Since

$$S(T) = S(t)\ e^{-\frac{\sigma_s^2}{2}(T-t) + \sigma_S[W^s(T) - W^s(t)]} \tag{5}$$

Using Eqs. (4) and (5), we get

$$\begin{aligned} e^{-\int_t^T r_d(s)\ ds}\ S(T) &= P(t,T)\ e^{-\frac{\sigma_d^2}{2} \int_t^T b_d^2(s,T)\ ds - \sigma_d \int_t^T b_d(s,T)\ dW^d(s)}\ . \\ &\quad S(t)\ e^{-\frac{\sigma_s^2}{2}(T-t) + \sigma_s[W^s(T) - W^s(t)]} \end{aligned}$$

Hence

$$\begin{aligned} E^Q\left[e^{-\int_t^T r_d(s)\ ds}\ S(T)\middle|\mathcal{F}(t)\right] &= \\ E^Q\left[S(t)\ P(t,T)\ e^{-\frac{\sigma_d^2}{2} \int_t^T b_d^2(s,T)\ ds - \sigma_d \int_t^T b_d(s,T)\ dW^d(s) - \frac{\sigma_s^2}{2}(T-t) + \sigma_S[W^s(T) - W^s(t)]}\right] \\ &= S(t)\ P(t,T)\ e^{-\rho^{s,d}\sigma_s\sigma_d \int_t^T b_d(s,T)\ ds} \end{aligned}$$

Hence

$$\frac{E^Q\left[e^{-\int_t^T r_d(s)\ ds}\ S(T)\middle|\mathcal{F}(t)\right]}{P(t,T)} = S(t)\ e^{-\rho^{s,d}\sigma_s\sigma_d\int_t^T b_d(s,T)\ ds}$$

Thus the use of stochastic rate introduces an extra term in the stock's price

$$e^{-\rho^{s,d}\sigma_s\sigma_d\int_t^T b_d(s,T)\ ds}$$

Hence the convexity adjustment is

$$CA = e^{-\rho^{s,d}\sigma_s\sigma_d\int_t^T b_d(s,T)\ ds} - 1$$

For deterministic rates, we have

$$\sigma_d = 0$$

Therefore,

$$CA = e^0 - 1 = 0$$

As expected, the convexity adjustment is zero for deterministic rates.

B. Calculation In T-Forward Measure

The geometric Brownian motion in the risk-neutral world is written as

$$dS(t) = r(t)S(t)\ dt + \sigma_s S(t)\ dW^s(t)$$

The corresponding Brownian motion in T-forward measure can be written as

$$dW^{s,T}(u) = dW^s(u) + \rho^{s,d}\sigma_s\sigma_P(t,T)\ dt$$

where $\rho^{s,d}$ is the correlation between asset S and rate r_d and $\sigma_P(t,T)$ is the bond volatility and is given by

$$\sigma_P(t,T) = \sigma_d\ b_d(t,T)$$

Here $\sigma_P(t,T)$ is the HJM bond volatility and it follows from the equation

$$dP(t,T) = r(t)P(t,T)\ dt - \sigma_P(t,T)P(t,T)\ dW^d(t)$$

To make connection with the short rate process seen earlier, we use the result for bond price under Hull-White model

$$P(t,T) = e^{-r_d(t)C(t,T)-A(t,T)}$$

where

$$C(t,T) = \int_t^T b_d(s,T)\ ds$$

This implies that

$$dP(t,T) = r(t)P(t,T)\ dt - \sigma_P(t,T)P(t,T)\ dW^d(t)$$

We can now write the dynamics for the driftless asset

$$dS(t) = \sigma_s\ S(t)\ dW^s(t)$$

In the T-forward measure

$$dS(t) = -\rho^{s,d}\sigma_s\sigma_P(t,T)\ dt + \sigma_s\ dW^{s,T}(t)$$

Therefore,

$$S(T) = S(t)\ e^{-\rho^{s,d}\sigma_s \int_t^T \sigma_P(s,T)\ ds - \frac{1}{2}\sigma_s^2(T-t) + \int_t^T \sigma_s\ dW^{s,T}(s)}$$

Now

$$\sigma_P(t,T) = \sigma_d b_d(t,T)$$

Hence

$$E^T[S(T)] = S(t)\ e^{-\rho^{s,d}\sigma_s \int_t^T \sigma_P(s,T)\ ds}$$

This is same as derived under risk-neutral measure.

VI. SUMMARY

We demonstrated that the convexity adjustment of the forward on driftless asset depends on the implied volatility of the asset and interest rate and correlation ρ between the interest rate and asset price as well as time to maturity of the trade. We also demonstrated that the convexity adjustment is zero for deterministic rates. Furthermore, the convexity adjustment is zero when the asset is not driftless with mean rate of return r. In previous article we demonstrated that the convexity adjustment occurs when an interest rate is paid out in the ”wrong” currency [7]. Note that in all cases, we assume that the asset return follow geometric Brownian motion. The next article will provide the calculation for asset return following the arithmetic Brownian motion (ABM).

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