A Gentle Introduction to Quanto

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Abstract

An attempt have been made to explain the mathematics underlying the Quanto effect in this essay. The convexity adjustment manifests when an interest rate is paid out at the ”wrong” time or in the ”wrong” currency [1]. Quanto adjustment also called as convexity adjustment has been shown to result from changes in the numeraire from foreign to domestic risk-neutral measure. The convexity adjustment also results from higher order derivative in stochastic calculations due to the use of Ito’s calculus. Note that the convexity adjustments are quantified in terms of the variance of the underlying process, it follows that implied volatility and time to maturity are its primary determinants. Hence the convexity adjustment increases with the implied volatility and time to maturity of the underlying. The convexity adjustment for the forward on driftless asset will be presented in the next article in both the bond numeraire (T-forward measure) and money market numeraire.

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I. INTRODUCTION

A quanto also known as quantity-adjusting option is an option denominated in a currency other than the currency in which the underlying asset is traded [1–9]. The quanto product converts underlying asset prices into units of the payoff currency by applying a fixed exchange rate. The quanto is traded for risk management purpose, particularly when dealing with exposure to foreign exchange rates. For instance, an investor who wants to enter in a foreign underlying asset option strategy but is just interested in the percentage return paid in their home currency without any exposure to international exchange rate risk. The payment of the quanto strategy will simply be this FX rate multiplied by the payout of the normal strategy. The FX rate will be locked to the rate in effect at the moment the trade is initiated. The payment is made in a different currency than the one used to compute the cashflows from the underlying. The article is structured as follows: A general introduction to convexity is provided in the II. Section III discusses the foreign and domestic measure. Derivation of stock's drift in domestic measure is also provided in this section. Stock drift is derived using the multivariate Girsanov theorem in Section IV. Section V discusses the partial differential equation approach. Some of the quanto products are discussed in Section VI. Finally, we summarise the results in Section VII.

II. CONVEXITY

In this section we will briefly discuss the convex function. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

where $\lambda \in [0, 1]$.

A. Examples of Convex Function

- e^{ax}
- $-\ln(x)$

Note that if $f(x)$ has a second derivative in $[a, b]$, then a necessary and sufficient condition for it to be convex on that interval is that the second derivative $f^{''}(x) \geq 0$ for all x in $[a, b]$.

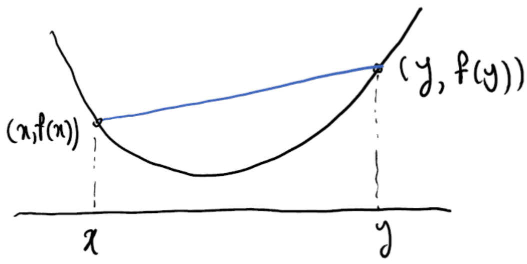

FIG. 1: Example of a Convex Function. Geometrically, the line segment between any two points on the graph of the function lies above the graph between the two points.

As an example let us consider Weiner process $W(t)$ and a function $f = f(W(t))$. Then using the Itô-Doeblin formula [4], we get

$$df(W(t)) = \frac{\partial f(W(t))}{\partial W(t)}\ dW(t) + \frac{1}{2}\frac{\partial^2 f(W(t))}{\partial W(t)^2}(dW(t))^2 + \text{Small Error}$$

Note that the second order term $\frac{1}{2}f''(W(t)(dW(t))^2$ capture the curvature of the function. Let

$$f(W(t)) = W^2(t)$$

Then

$$df(W(t)) = 2W(t)\ dW(t) + dt + \mathcal{O}\left(t^{3/2}\right)$$

B. Convexity of Stock's Price – Adjustment Due to Itô's Lemma

Let the stochastic differential equation for the asset is assumed to be of the form

$$dS(t)\ =\ r(t)\ dt + S(t)\ \sigma_s\ dW^s(t)$$

Let us define

$$f(S(t)) = \ln(S(t))$$

Again using the Itô-Doeblin formula [4], we get

$$df(S(t)) = \frac{\partial f(S(t))}{\partial S(t)}\ dS(t) + \frac{1}{2}\frac{\partial^2 f(S(t))}{\partial S(t)^2}(dS(t))^2 + \text{Small Error}$$

From here we get

$$df(S(t)) = r\ dt + \sigma_s\ dW^s(t) - \frac{1}{2}\sigma_s^2\ dt + \mathcal{O}\left(t^{3/2}\right)$$

Integrating this equation, we obtain

$$S(t) = S(0)\ e^{\left(r-\frac{\sigma_s^2}{2}\right)t+\sigma_s\ W^s(t)}$$

Since the Brownian motion $W^s(t)$ is martingale, the function

$$S(0)\ e^{\sigma_s W^s(t)}$$

is not a martingale but

$$S(0)\ e^{\sigma_s W^s(t)-\frac{1}{2}\sigma_s^2 t}$$

is a martingale and hence the mean rate of return of asset is not equal to $r - \frac{\sigma^2}{2}$, it is equal to r. Note that the convexity of the function $e^{\sigma x}$ imparts an upward drift to $S(0)\ e^{\sigma_s W^s(t)}$. The correction term is

$$S(0)\ e^{-\frac{\sigma_s^2}{2}t}$$

Note that this equation can also be written as

$$S(0)\ e^{-\frac{\sigma_s \times \sigma_s \times \rho^{s,s}}{2}\ t}$$

where $\rho^{s,s} = 1$, i.e., the correlation of stock return with itself. Thus the convexity correction depends on the product of implied volatility of the underlying, the correlation between the underlying and the time to maturity t. The convexity adjustment in general depends on these three factors. If $r = 0$, then

$$dS(t)\ =\ S(t)\ \sigma_s\ dW^s(t)$$

Then

$$S(t) = S(0)\ e^{\sigma_s\ W^s(t)-\frac{\sigma_s^2}{2}t}$$

is a martingale. Let us also consider the case of arithmetic Brownian motion

$$dS(t) = \mu\ dt + \sigma_s\ dW^s(t)$$

Note that there is no convexity adjustment in this case because we do not need a transformation like in the previous case.

III. QUANTITY-ADJUSTING OPTION

A. Different Cases

The following paragraphs cover the three different cases:

1. **Conversion:** A Foreign Equity Option Converted to Domestic Currency is given by

$$\text{Payoff} \ = \ X_T \max\left(S_T - K_f,\ 0\right)$$

where S_T is the asset price in foreign currency, K_f is the strike price in foreign currency and X_T is the exchange rate.

2. **Quanto:** The payoff of a quanto call option is given by

$$\text{Payoff} \ = \ \overline{X} \max\left(S_T - K_f,\ 0\right) = \max\left(\overline{X} S_T - K_d,\ 0\right)$$

where S_T is the asset price in foreign currency, K_f is the strike price in foreign currency and $\overline{X}$ is some pre-determined fixed exchange rate.

3. **Struck:** The struck is an option strategy on a foreign stock with strike in domestic currency and the payout of this option in the domestic currency.

$$\text{Payoff} \ = \ \max\left(X_T S_T - K_d,\ 0\right)$$

Please note that the Struck lead to squared payoffs. Also, the holder of a composite option has exposure to the exchange rate.

B. Stock Price Under Different Measure

We must estimate the stock price drift in domestic measure in order to price the quanto.

1. Stock Price in Foreign Measure

We know that under $\mathbb{P}$ measure

$$dS_t = \mu_s S_t \ dt + \sigma_s S_t \ dW_t^s$$

From the perspective of a foreign investor with foreign cash account as numeraire, the foreign asset's risk neutral drift in this equation would be $\mu_{s_t} = r^f$ and hence in foreign risk neutral measure, this equation can be written as

$$dS_t = r^f \ S_t \ dt + \sigma_s S_t \ dW_t^{s^f}$$

However, this is not true for domestic investor, since S_t is not the domestic price of a traded asset nor is the foreign cash account a domestic numeraire. In fact it is $X_t S_t$ that is the domestic price of a traded asset.

2. Stock Price in Domestic Measure

Let the SDE for S_t under $\mathbb{Q}^d$ measure be

$$dS_t = \mu_s^d S_t \ dt + \sigma_s S_t \ dW_t^{s^d}$$

where μ_s^d is unknown.

C. Foreign Exchange Rate under Domestic Risk-Neutral Measure

Consider a FX rate X_t, say EUR/USD. The value of the foreign money market account in domestic currency is $B_t^f X_t$, and its discounted value in domestic currency is

$$\widetilde{X_t} = \frac{X_t B_t^f}{B_t^d}$$

Here, $X_t B^f$ denotes the foreign risk-free asset quoted in domestic currency and

$$\begin{aligned} dX_t &= \mu_t X_t \ dt + \sigma_t X_t \ dW_t^x \\ dB_t^d &= r_t^d B_t^d \ dt \\ dB_t^f &= r_t^f B_t^f \ dt \end{aligned}$$

We will show using Girsanov's theorem, that by changing the measure $\mathbb{P}$ to an equivalent domestic risk-neutral measure $\mathbb{Q}^d$ that $\widetilde{X}_t$ is a $\mathbb{Q}^d$-martingale. Using Itô's lemma

$$\begin{aligned}
d\widetilde{X}_t &= \frac{\partial \widetilde{X}_t}{\partial X_t}\, dX_t + \frac{\partial \widetilde{X}_t}{\partial B_t^d}\, dB_t^d + \frac{\partial \widetilde{X}_t}{\partial B_t^f}\, dB_t^f + \frac{1}{2}\frac{\partial^2 \widetilde{X}_t}{\partial X_t^2}\, (dX_t)^2 + \cdots \\
&= \frac{B_t^f}{B_t^d}\left[\mu_t X_t\, dt + \sigma_t X_t\, dW_t^x\right] - \frac{X_t B_t^f}{(B_t^d)^2} r_t^d B_t^d\, dt + \frac{X_t}{B_t^d} r_t^f B_t^f\, dt \\
&= \left[\mu_t + r_t^f - r_t^d\right] \widetilde{X}_t\, dt + \sigma_t \widetilde{X}_t\, dW_t^x \\
&= \sigma_t \widetilde{X}_t\, d\widetilde{W}_t^d
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{W}_t^d &= W_t^x + \int_0^t \lambda_u\, du \\
\lambda_u &= \frac{\mu_t + r_t^f - r_t^d}{\sigma_t}
\end{aligned}$$

From Girsanov's theorem, we know that there exist an equivalent risk-neutral measure $\mathbb{Q}^d$ on filtration $\mathcal{F}_s$, $0 \le s \le t$ defined by Radon-Nikodỳm derivative

$$Z_s = e^{-\int_0^s \lambda_u dW_u^x - \frac{1}{2}\int_0^s \lambda_u^2\ du}$$

so that $\widetilde{W}_t^d$ is a $\mathbb{Q}^d$-standard Weiner process. Substituting

$$dW_t^x = d\widetilde{W}_t^d - \lambda_t\ dt$$

into

$$dX_t = \mu_t X_t\ dt + \sigma_t X_t\ dW_t^x$$

we get

$$dX_t = \left(r_t^d - r_t^f\right) X_t\ dt + \sigma_t X_t\ d\widetilde{W}_t^d \tag{1}$$

Note that if one consider exchange rate as asset, then it is dividend paying asset.

Siegel's Exchange Rate Paradox

We know that the mean rate of return of exchange rate in domestic measure is $r_t^d - r_t^f$ (see Eq. (1)). When seen from a foreign prespective, the exchange rate is $\frac{1}{X}$. Also, one might expect the mean rate of change of $\frac{1}{X}$ to be $r_t^f - r_t^d$. Let

$$f\left(\widetilde{X}_t\right) = \frac{1}{\widetilde{X}_t}$$

Then

$$d\left(\frac{1}{\widetilde{X}_t}\right) = f'\left(\widetilde{X}_t\right) d\widetilde{X}_t + \frac{1}{2} f''\left(\widetilde{X}_t\right) d\widetilde{X}_t^2$$
$$= \frac{1}{\widetilde{X}_t}\left[\left(r_t^f - r_t^d + \sigma_t^2\right) dt - \sigma_t \, d\widetilde{W}_t^d\right] \qquad (2)$$

Thus the mean rate of return under domestic measure is

$$r_t^f - r_t^d + \sigma_t^2$$

Note that the extra term σ_t^2 arises due to the convexity of the function $f\left(\widetilde{X}_t\right) = \frac{1}{\widetilde{X}_t}$. Thus

$$f'\left(\widetilde{X}_t\right) = -\frac{1}{\widetilde{X}_t^2} \quad \text{and} \quad f''\left(\widetilde{X}\right) = \frac{2}{\widetilde{X}_t^3}$$

If we switch to the foreign risk-neutral measure, which is the proper one for pricing derivative security in the foreign currency, the imbalance caused by the convexity of $f\left(\widetilde{X}\right) = \frac{1}{\widetilde{X}}$ is resolved. See Ref. [4] for detailed discussion.

D. Foreign Exchange Rate under Foreign Risk-Neutral Measure

We know that

$$d\left(\frac{1}{\widetilde{X}_t}\right) = \frac{1}{\widetilde{X}_t}\left[\left(r_t^f - r_t^d + \sigma_t^2\right) dt - \sigma_t \, d\widetilde{W}_t^d\right] \qquad (3)$$

Let

$$d\widetilde{W}_t^f = d\widetilde{W}_t^d - \sigma_t \, dt$$

In terms of $\widetilde{W}_t^f$, we may rewrite Eq. (3) as

$$d\left(\frac{1}{\widetilde{X}_t}\right) = \frac{1}{\widetilde{X}_t}\left[\left(r_t^f - r_t^d\right) dt - \sigma_t \, d\widetilde{W}_t^f\right]$$

As expected, under the foreign risk neutral measure, the mean rate of change for $\frac{1}{\widetilde{X}}$ is $r_t^f - r_t^d$.

E. Drift Estimation Using Replicating Portfolio

The SDE for the non-dividend paying stock price quoted in foreign currency under $\mathbb{P}$ measure is given by [3]

$$dS_t = \mu_s S_t \, dt + \sigma_s S_t \, dW_t^s$$

We also have FX rate SDE as

$$dX_t = \mu_x X_t \ dt + \sigma_x S_t \ dW_t^x$$

and

$$dW_t^s \cdot dW_t^x = \rho \ dt$$

1. Estimation of Stock Drift in Domestic Measure

We know that under $\mathbb{Q}^d$ measure

$$dX_t \ = \ \left(r^d - r^f\right) X_t \ dt + \sigma_x X_t \ dW_t^{x^d}$$

We will prove that

$$\frac{d(X_t S_t)}{X_t S_t} \ = \ r^d \ dt + \sqrt{\sigma_x^2 + 2\rho\sigma_s\sigma_x + \sigma_s^2} \ d\widetilde{W}_t^{xs} \tag{4}$$

Proof: We know that

$$\begin{aligned} d(X_t S_t) \ &= \ X_t \ dS_t + S_t \ dX_t + dX_t \cdot dS_t \\ &= \ \mu_s^d X_t S_t \ dt + \sigma_s X_t S_t \ dW_t^{s^d} + S_t X_t \left(r^d - r^f\right) dt + S_t X_t \sigma_x \ dW_t^\rho X^d + \sigma_s \sigma_x \ dt \\ &= \ X_t S_t \left[\mu_s^d + r^d - r^f + \rho\sigma_s\sigma_x\right] dt + \sigma_s X_t S_t \ dW_t^{s^d} + S_t X_t \sigma_x \ dW_t^{x^d} \end{aligned}$$

where

$$\rho \ dt = dW_t^{x^d} \cdot dW_t^{s^d}$$

Thus

$$\frac{d(X_t S_t)}{X_t S_t} = \left[\mu_s^d + r^d - r^f + \rho\sigma_s\sigma_x\right] dt + \sigma_s \ dW_t^{s^d} + \sigma_x \ dW_t^{x^d}$$

Comparing with (4), we get

$$\begin{aligned} \mu_s^d \ &= \ r^f - \rho\sigma_x\sigma_s \\ d\widetilde{W}_t^{xs} \ &= \ \frac{\sigma_x dW_t^{x^d} + \sigma_s dW_t^{s^d}}{\sqrt{\sigma_x^2 + 2\rho\sigma_s\sigma_x + \sigma_s^2}} \end{aligned}$$

Proof of (4): We have

$$\begin{aligned} \frac{d(X_t S_t)}{X_t S_t} \ &= \ \left[\mu_x + \mu_s + \rho\sigma_s\sigma_x\right] dt + \sigma_s \ dW_t^s + \sigma_x \ dW_t^x \\ &= \ \left[\mu_x + \mu_s + \rho\sigma_s\sigma_x\right] dt + \sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x} \ dW_t^{xs} \\ &= \ \mu \ dt + \sigma \ dW_t^{xs} \end{aligned}$$

where

$$
\begin{aligned}
\mu &= \mu_x + \mu_s + \rho\sigma_s\sigma_x \\
\sigma &= \sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x} \\
dW_t^{xs} &= \frac{\sigma_s\ dW_t^s + \sigma_x\ dW_t^x}{\sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x}}
\end{aligned}
$$

Here W_t^{xs} is a $\mathbb{P}$-standard Weiner process.

2. Replication

At time t, we let the portfolio Π_t be valued as

$$\Pi_t = \phi_t U_t + \psi_t B_t^d$$

where ϕ_t and ψ_t are the units invested in $U_t = X_t S_t$ and the risk-free asset B_t^d, respectively.

$$
\begin{aligned}
d\Pi_t &= \phi_t \left[(\mu_x + \mu_s + \rho\sigma_s\sigma_x)\, U_t\ dt + \sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x}\ U_t\ dW_t^{xs} \right] + \psi_t r^d B_t^d\ dt \\
&= r^d \Pi_t\ dt + \phi_t \left[\left(\mu_x + \mu_s + \rho\sigma_s\sigma_x - r^d \right) U_t\ dt + \sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x}\ U_t\ dW_t^{xs} \right] \\
&= r^d \Pi_t\ dt + \phi_t \sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x}\ U_t\ d\widetilde{W}_t^{xs}
\end{aligned}
$$

where

$$\widetilde{W}_t^{xs} = \lambda\ t + W_t^{xs}$$

with

$$\lambda = \frac{\mu_x + \mu_s + \rho\sigma_s\sigma_x}{\sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x}}$$

Finally, by substituting

$$dW_t^{xs} = d\widetilde{W}_t^{xs} - \lambda\ dt$$

into

$$\frac{d(X_t S_t)}{X_t S_t} = (\mu_x + \mu_s + \rho\sigma_s\sigma_x)\, dt + \sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x}\ dW_t^{xs}$$

The stock price diffusion process under the risk-neutral measure $\mathbb{Q}^d$ becomes

$$\frac{d(X_t S_t)}{X_t S_t} = r^d\ dt + \sqrt{\sigma_s^2 + \sigma_x^2 + \rho\sigma_s\sigma_x}\ d\widetilde{W}_t^{xs}$$

Thus, under the domestic risk-neutral measure $\mathbb{Q}^d$

$$\frac{dS_t}{S_t} = \left(r^f - \rho\ \sigma_x\sigma_s \right) dt + \sigma_s\ dW_t^{s^d}$$

where r^f is risk free rate in foreign currency, σ_x is volatility of FX rate, σ_s is volatility of stock price and ρ is correlation between FX rate and stock price.

Comment on Drift Term

The drift term in domestic measure from above equation is given by [2]

$$r^f - \rho\ \sigma_x \sigma_s \tag{5}$$

where r^f is risk free rate in foreign currency, σ_x is volatility of FX rate, σ_s is volatility of stock price and ρ is correlation between FX rate and stock price.

A. Effect of Correlation

Let us consider an US investor who expects that British stock XYZ share price will increase in one year. The US investor long a 1Y ATM call on British stock XYZ and wants to get his return in USD and not in GBP. Assume that XYZ stock price is £ 5,

$$\text{FX Rate} = \frac{\text{GBP}}{\text{USD}} = 2$$

Thus

$$1 \text{ GBP} = 2 \text{ USD}$$

Let us assume that in one year time, the XYZ share price is worth £ 5.50. Then for plain case

$$\text{Payoff} = \max(S_T - K) = £\ 0.5$$

The Payoff of quanto call option is given by

$$\text{Payoff} = \overline{X} \times \max(S_T - K) = \$\ 0.5 \times 2 = \$\ 1$$

regardless of the change in exchange rate. Let us consider two different cases:

1. **Positive Correlation:** Assume that at expiry the exchange rate is

$$\text{FX Rate} = \frac{\text{GBP}}{\text{USD}} = 3.0$$

Then the Payoff of without quanto call option would be

$$\text{Payoff} = X_t \times \max(S_T - K) = \$ \ 0.5 \times 3 = \$ \ 1.5$$

In this case, the GBP gets more valuable against the USD, XYZ price tends to increase. The investor would then have been better of with a vanilla call. He is therefore short this correlation.

2. **Negative Correlation:** Assume that at expiry the exchange rate is

$$\text{FX Rate} = \frac{\text{GBP}}{\text{USD}} = 1.5$$

Then the Payoff of without quanto call option would be

$$\text{Payoff} = X_t \times \max(S_T - K) = \$ \ 0.5 \times 1.5 = \$ \ 0.75$$

In this case, the GBP gets less valuable against the USD, XYZ price tends to increase. The investor would then would be better of with a quanto call. He is therefore long this correlation.

B. Effect of Volatility

1. FX volatility σ_x: Is the European investor long or short the FX volatility? We know that

$$\frac{dS_t}{S_t} = \left(r^f - \rho \ \sigma_x \sigma_s\right) dt + \sigma_s \ dW_t^{s^d}$$

In domestic currency (say USD), we know that the model for the change in stock price with a dividend yield equal to d is

$$\frac{dS_t}{S_t} = \left(r^d - d\right) dt + \sigma_s \ dW_t^{s^d}$$

Therefore the price of a quanto option can be derived from a normal option by making an adjustment to dividend yield

$$\frac{dS_t}{S_t} = \left(r^d - r^f + \rho \ \sigma_x \sigma_s\right) dt + \sigma_s \ dW_t^{s^d}$$

It is obvious from this equation that the FX volatility sensitivity is influenced by the correlation's sign. The holder of a quanto call (put) is long (short) FX volatility if the correlation is negative. The holder of a quanto call (put) is short (long) FX volatility if the correlation is positive. It also emphasises the obvious principle that the implied volatility of the underlying stock should be the same as the volatility utilised to price the quanto option.

2. The changes in FX rates have an impact on the delta hedge. The delta hedge financing will benefit if the price of the underlying stock doubles since one will be able to sell more shares at a higher price and earn more interest as a result. However, the delta hedge financing is actually unaffected if the correlation is such that a doubling in the price of the underlying stock causes a halve of the EUR value versus the GBP.

IV. STOCK DRIFT USING MULTIVARIATE GIRSANOV THEOREM

We saw in the previous section how the change in measure is used to estimate stock drift in domestic measures. In this section, we will estimate the drift of stock price in domestic measure using the multivariate Girsanov theorem [6].

A. Girsanov Theorem in One Dimension

Let $\{W_t : 0 \leq t \leq T\}$ be a Weiner process on $\Omega, \mathcal{F}, P$. Suppose $\Theta(t)$ is an adapted process and consider

$$Z(t) = e^{-\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t ||\Theta(u)||^2 \ du}$$

If

$$E^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T ||\Theta(u)||^2 \ du}\right] < \infty$$

Then Z_t is a $\mathbb{P}$-martingale for $0 \leq t \leq T$. Furthermore,

$$E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle| \mathcal{F}_t\right] = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t$$

Then

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u) \ du$$

is a Q standard Weiner process. Also

$$E^{\mathbb{Q}}[X_t] = E^{\mathbb{P}}\left[X_t \left(\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}\right)\right]$$

and hence

$$\int_A X_t \, d\mathbb{Q} = \int_A X_t \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} \right) d\mathbb{P}, \qquad A \in \mathcal{F}_t$$

Theorem: Let us consider a probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ and a nonnegative random variable Z satisfying

$$E^{\mathbb{P}}[Z] = 1$$

Then defined a new probability measure $\mathbb{Q}$ by the formula

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

The expectation of a random variable X in the measure $\mathbb{P}$ and $\mathbb{Q}$ are related by

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[XZ]$$

Here Z is the Radon-Nikodým derivative of $\mathbb{P}$ with respect to $\mathbb{Q}$, and we write

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

B. Multivariate Girsanov Theorem

Let $\mathbf{Z}$ be exponential martingale under probability measure $\mathbb{P}$. Therefore

$$\frac{d\mathbf{Z}}{\mathbf{Z}} = \boldsymbol{\Theta}^T \, d\mathbf{W}(t)$$

A new probability measure $\mathbb{Q}$ is defined using $\mathbf{Z}$ as Radon-Nikodỳm theorem as

$$E^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] = \mathbf{Z}(\mathbf{t})$$

Under the new probability measure $\mathbb{Q}$, the process is defined by

$$\widetilde{\mathbf{W}}(t) = \mathbf{W}(t) + \int_0^t \boldsymbol{\Theta}(u) \, du$$

As an example, let us consider

$$\begin{aligned} d\mathbf{X} &= \boldsymbol{\mu}_{\mathbf{X}} \mathbf{X} \, dt + \mathbf{X} \boldsymbol{\Sigma}_{\mathbf{X}}^T \, d\mathbf{W}^{\mathbf{X}}(t) \\ &= \boldsymbol{\mu}_{\mathbf{X}} \mathbf{X} \, dt + \mathbf{X} \boldsymbol{\Sigma}_{\mathbf{X}}^T \left[d\widetilde{\mathbf{W}}(t) + \boldsymbol{\Theta}(t) \, dt \right] \\ &= \mathbf{X} \left[\boldsymbol{\mu}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{X}}^T \, \boldsymbol{\Theta}(t) \right] dt + \mathbf{X} \boldsymbol{\Sigma}_{\mathbf{X}}^T d\widetilde{\mathbf{W}}^{\mathbf{X}}(t) \\ &= \underbrace{\mathbf{X} \widetilde{\boldsymbol{\mu}}_{\mathbf{X}}}_{\text{New Drift}} \, dt + \mathbf{X} \boldsymbol{\Sigma}_{\mathbf{X}}^T \, d\widetilde{\mathbf{W}}^{\mathbf{X}}(t) \end{aligned}$$

Theorem: The new drift in the new measure is given by the following relation

$$\widetilde{\boldsymbol{\mu}}_{\mathbf{X}} = \boldsymbol{\mu}_{\mathbf{X}} + \frac{d}{dt}\langle \ln \mathbf{X},\ \ \ln \mathbf{Z}\rangle$$

In other words

$$\boldsymbol{\Sigma}_{\mathbf{X}}^{T}\ \boldsymbol{\Theta}(t) = \frac{d}{dt}\langle \ln \mathbf{X},\ \ \ln \mathbf{Z}\rangle$$

Proof: Since

$$\begin{aligned} d\langle \ln \mathbf{X},\ \ \ln \mathbf{Z}\rangle &= \left\langle \frac{d\mathbf{X}}{X},\ \frac{d\mathbf{Z}}{Z}\right\rangle \\ &= \left\langle \boldsymbol{\Sigma}_{\mathbf{X}}^{T}\ d\mathbf{W}_{\mathbf{X}}(t),\ \ \boldsymbol{\Theta}^{T}(t) d\mathbf{W}(t)\right\rangle \\ &= \boldsymbol{\Sigma}_{\mathbf{X}}^{T}\boldsymbol{\Theta}(t)\ dt \end{aligned}$$

where

$$\langle \mathbf{A},\ \mathbf{B}\rangle = \mathbf{A}^{T}\mathbf{B}$$

and

$$\left\langle \mathbf{A}^{T}\ d\mathbf{W}(t),\ \ \mathbf{B}^{T}\ d\mathbf{W}(t)\right\rangle = \mathbf{A}^{T}\mathbf{B}\ dt$$

C. Drift Estimation Of Quanto Using Multivariate Girsanov Theorem

We know that the SDE for FX in domestic measure is

$$dX_t = \left(r^d - r^f\right) X_t\ dt + \sigma_x X_t\ dW_x^d(t)$$

Also we know that

$$\frac{X_t S_t}{B_t^d} = E^{\mathbb{Q}^d}\left[\left.\frac{X_T S_T}{B_T^d}\right| \mathcal{F}_t\right]$$

This in turn implies that

$$S_t = e^{-r^d\tau} E^{\mathbb{Q}^d}\left[\left.\frac{X_T S_T}{X_t}\right| \mathcal{F}_t\right]$$

Also

$$\frac{S_t}{B_t^f} = E^{\mathbb{Q}^f}\left[\left.\frac{S_T}{B_T^f}\right| \mathcal{F}_t\right]$$

Hence

$$S_t = e^{-r^f\tau} E^{\mathbb{Q}^f}[S_T|\mathcal{F}_t]$$

Comparing the two equations for S_t, we get

$$e^{-r^f\tau} E^{\mathbb{Q}^f}[S_T|\mathcal{F}_t] = e^{-r^d\tau} E^{\mathbb{Q}^d}\left[\left.\frac{X_T S_T}{X_t}\right| \mathcal{F}_t\right]$$

This can be written as

$$\begin{aligned} E^{\mathbb{Q}^f}[S_T|\mathcal{F}_t] &= e^{-\left(r^d-r^f\right)\tau} E^{\mathbb{Q}^d}\left[\left.\frac{X_T S_T}{B_T^d}\right|\mathcal{F}_t\right] \\ &= e^{-\left(r^d-r^f\right)\tau} E^{\mathbb{Q}^f}\left[\left.\frac{X_T S_T}{X_t}\cdot\frac{d\mathbb{Q}^d}{d\mathbb{Q}^f}\right|\mathcal{F}_t\right] \end{aligned}$$

Now

$$X_T = e^{\left(r^d-r^f\right)T+\sigma_x W_T}$$

and

$$\frac{X_T}{X_t} = e^{\left(r^d-r^f\right)\tau+\sigma_x(W_T-W_t)}$$

Hence

$$\frac{d\mathbb{Q}^d}{d\mathbb{Q}^f} = \frac{X_t}{X_T}\cdot e^{\left(r^d-r^f\right)\tau} = e^{-\sigma_x(W_T-W_t)}$$

Similarly, we can show that

$$\frac{d\mathbb{Q}^f}{d\mathbb{Q}^d} = \frac{X_T}{X_t}\cdot e^{-\left(r^d-r^f\right)\tau} = e^{\sigma_x(W_T-W_t)}$$

In the foreign measure, we can write

$$\frac{dS}{S} = r^f\ dt + \sigma_s\ dW_s^f(t)$$

Then going from foreign to domestic measure, we can write

$$Z = \frac{d\mathbb{Q}^d}{d\mathbb{Q}^f}$$

and hence

$$Z(t) = E^{\mathbb{Q}^f}\left[\frac{d\mathbb{Q}^d}{d\mathbb{Q}^f}\right]$$

From here we get

$$\frac{dZ}{Z} = -\sigma_x dW_x^f(t)$$

When going from foreign to domestic measure, we need to find the new drift

$$\begin{aligned} \widetilde{\mu}(t) &= \left\langle \frac{dS}{S}, \frac{dZ}{Z}\right\rangle \\ &= \left\langle \sigma_s\ dW_S^f(t),\ \ -\sigma_s dW_x^f(t)\right\rangle \\ &= -\sigma_s\sigma_x\rho\ dt \end{aligned}$$

Hence

$$\frac{dS}{S} = \left(r^f - \rho\sigma_s\sigma_x\right) dt + \sigma_s dW_s^d(t)$$

If we compare this equation with the Black-Scholes equation with a constant dividend yield we see that pricing the quanto is equivalent to using a dividend yield of

$$r^d - r^f + \rho\sigma_s\sigma_x$$

Note that there is an adjustment to a dividend yield. This yield depends on the volatility of the exchange rate and the correlation between the underlying and the exchange rate.

V. A PDE FORMULATION OF QUANTO

Define X to be the EUR/USD exchange rate (number of Euros per Dollar) and S is the level of the Euro index [9]. We assume that they satisfy

$$\begin{aligned} dX_t &= \mu_x X_t \, dt + \sigma_x X_t \, dW_t^x \\ dS_t &= \mu_s S_t \, dt + \sigma_s S_t \, dW_t^s \end{aligned}$$

with a correlation coefficient ρ between them. For hedging purpose, we construct a portfolio that includes the Euro and the Euro index

$$\Pi = V(X_t, S_t, t) - \Delta_x X - \Delta_s X S$$

Note that every term in this equation is measured in domestic currency dollar. We short Δ_x number of Euro. Note that $-\Delta_x X$ is the dollar value of that Euro. Similarly, with the term $\Delta_s XS$ we have converted the Euro-denominated index S into dollars. Again, Δ_s is the number of the index we hold short. The change in the value of the portfolio is due to the change in the value of its components and the interest received on the Euro:

$$\begin{aligned} d\Pi = & \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_x^2 X_t^2 \frac{\partial^2 V}{\partial X_t^2} + \rho\sigma_s\sigma_x X_t S_t \frac{\partial^2 V}{\partial X_t \partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}\right. \\ & \left. -\rho\sigma_s\sigma_x \Delta_s S_t X_t - r^f \Delta_x X_t\right) dt + \left(\frac{\partial V}{\partial X_t} - \Delta_x - \Delta_s S_t\right) dX_t + \\ & \left(\frac{\partial V}{\partial S_t} - \Delta_s X_t\right) dS_t \end{aligned}$$

We now choose

$$\Delta_x = \frac{\partial V}{\partial X_t} - \frac{S_t}{X_t}\frac{\partial V}{\partial S_t}$$
$$\Delta_s = \frac{1}{X_t}\frac{\partial V}{\partial S_t}$$

to eliminate the risk in the portfolio. Setting the return on this riskless portfolio equal to the US risk-free rate of interest r^d, since Π is measured entirely in dollars, yields

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_x^2 X_t^2 \frac{\partial^2 V}{\partial X_t^2} + \rho\sigma_s\sigma_x X_t S_t \frac{\partial^2 V}{\partial X_t \partial S_t} + \frac{1}{2}\sigma_s^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} +$$
$$X_t \frac{\partial V}{\partial X_t}\left(r^d - r^f\right) + S_t \frac{\partial V}{\partial S_t}\left(r^f - \rho\sigma_s\sigma_x\right) - r^d V = 0$$

This completes the formulation of the pricing equation. This equation is valid for both path-independent and path-dependent cases. For both cases we need to specify the boundary condition.

VI. QUANTO PRODUCTS

In this section we will discuss some of the quanto products and its pricing.

A. Currency Forward

Let F_t^T denote the time t price of a forward contract for delivery of the foreign currency at time T [8, 9]. Then since the initial value of the forward contract is 0, martingale pricing implies

$$0 = E^{\mathbb{Q}}\left[\left.\frac{F_t^T - X_T}{B_T}\right|\mathcal{F}_t\right]$$

which in turn implies

$$F_t^T = E^{\mathbb{Q}}\left[X_T|\mathcal{F}_t\right] = e^{\left(r^d - r^f\right)\tau} X_t$$

A forward FX rate F_t^T is usually quoted as a premium or discount to the spot rate X_t, via the forward points.

B. Quanto Forward

$\overline{X}S_T$ is exchanged at maturity T for K_d unit of domestic currency. If we enter this contract at time t, then $K_{t,d} = K_d$ is chosen so that the initial value of contract is zero. Using domestic cash account as numerare, we get

$$0 = E^{\mathbb{Q}^d}\left[e^{-r^d\tau}\left(\overline{X}S_T - K_{t,d}\right)\middle|\,\mathcal{F}_t\right]$$

From here we get

$$K_{t,d} = \overline{X}E^{\mathbb{Q}^d}\left[S_T|\,\mathcal{F}_t\right]$$

We need $\mathbb{Q}^d$ dynamics of S_t in order to compute $K_{t,d}$. The $\mathbb{Q}^d$ dynamics of S_t is given by

$$dS_t = \left(r^f - \rho\sigma_s\sigma_x\right)S_d\ dt + \sigma S_t\ dW_t^d$$

Hence

$$K_{t,d} = \overline{X}S_t\ e^{\left(r^f - \rho\sigma_s\sigma_x\right)\tau}$$

This is the fair value of Quanto forward. Note that the convexity adjustment in this case is $e^{-\rho\sigma_s\sigma_x\tau} - 1$.

C. Quanto Call Option

Consider a European call option on foreign asset, with strike K_d is set in the domestic currency and the value of the foreign asset being converted to domestic currency at fixed exchange rate $\overline{X}$. The payoff of quanto call is given by

$$\text{Payoff} = \max\left(\overline{X}\ S_T - K_d,\ 0\right)$$

where S_t is foreign asset. The value of European call option is given by Black-Scholes formula

$$V(K,\sigma) = e^{-r_{div}\tau}\ S_t\ N(d_1) - e^{-r^d\tau}\ K\ N(d_2)$$

where r_{div} is dividend on stock and r^d is domestic interest rate. Since we know the dividend for quanto, we may rewrite the above formula for quanto call as

$$V_{\text{Quanto}}(K,\sigma) = e^{\left(r^f - r^d - \rho\sigma_s\sigma_x\right)\tau}\cdot S_t\overline{X}\cdot N(d_1) - e^{-r^d\tau}K_d\cdot N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{\overline{X}S_t}{K_d}\right) + \left(r^f - \rho\sigma_s\sigma_x + \frac{\sigma_s^2}{2}\right)\tau}{\sigma_s\sqrt{\tau}} \\ d_2 &= d_1 - \sigma_s\sqrt{\tau} \end{aligned}$$

VII. SUMMARY

We have provided a step-by-step guide for figuring out the stock price's drift in domestic measure in this article. Three alternative techniques are used to estimate the stock price's drift. Convexity adjustment result from switching from changing foreign to domestic measures. In other words, convexity adjustment manifests when an interest rate is paid out in the "wrong" currency [1]. The convexity adjustment for the forward on driftless asset will be provided in the next article.

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