Symmetric Duality in Multi-Objective Programming

Dr. Paras Bhatnagar
Deptt. of Applied Sciences & Humanities
College of Engineering & Technology
Moradabad

Dr. Mayank Pawar
Deptt. of Mathematics
Teerthankar Mahaveer University
Moradabad

Dr. V.P. Jain
Deptt. of Mathematics
Teerthankar Mahaveer University
Moradabad

1. Introduction

Dorn[6] introduced symmetric duality in nonlinear programming by defining a program and its dual to be symmetric if the dual of the dual is the original problem. In the past, the symmetric duality has been studied extensively in the literature, notable by Dantzig et al. [5], Mond [7] and Mond and Weir [9].

Recently, Weir and Mond [13] studied symmetric duality in the context of multi-objective programming by introducing a multi-objective analogue of the primal-dual pair presented in Mond[8]. Although the multi-objective primal dual pair constructed in [13] subsumes the single objective symmetric duality [7] as a special case, the construction of [13] seems to be somewhat restricted because the same parameter $\lambda \in R^p$ (vector multiplier corresponding to various objectives) is present in both primal and dual. Further, the proof of the main duality result [13] assumes this λ to be fixed in the dual problem.

The main aim of this part is to present a pair of multi-objective programming problem (P) and (D) with λ as variable in both programs and to establish symmetric duality by associating a vector-valued infinite game to this pair. Although this construction seems to be more natural than that of [13] as it does not require λ to be fixed in the dual problem, it lacks the weak duality theorem as illustrated in Section 3. However the case of single objective symmetric duality [7] is fully subsumed here as well, because (P) and (D) then reduce to the primal-dual pair of Dantzig et al.[5].

2. Preliminaries and statement of problems

Let R^n be an n-dimensional Euclidean space and R_+^n be its non-negative orthant. For $z, w \in R^n$, by $z \leq w$ we mean $z_i \leq w_i$ for all i, and $z \leq w$ means $z_i \leq w_i$ for all i and $z_s > w_s$, for at least on s, $1 \leq s \leq n$. By z<w, we mean $z_i < w_i$ for I . Let $e = (1,1,....1)^T \in R^p$ and $\wedge = \{\lambda \in R_+^p : \lambda^T e = 1\}$.

We now consider the vector-valued two-person zero-sum game $G : \{X, Y, K\}$.where

(i) $X = x \in R_+^m : p_k(x) \leq 0, k = 1,2,......,s$ is the space of strategies for player I.

(ii) $Y = y \in R_+^m : q_k(x) \geq 0, r = 1,2,.....,t$ is the space of strategies for player II.

(iii) $K : X \times Y \to R^p$ given by $K(X,Y) = (K_1(x,y), K_2(x,y),....,K_p(x,y))$, is the payoff to player I and $-K$ *(x,y)* is the payoff to player II.

In this presentation it is assumed that player I solves the "min-max problem" and player II solves the "max-min problem" in the sense of Definition 3 given below. Also the symbol "V-max" stands for vector maximization and V-minstands for vector minimisation.

The following definition will be needed in this sequel.

.DEFINITION 1. (Corley [1]): A point $(\bar{x},\bar{y}) \in X \times Y$ is said to be an equilibrium point of the game G if

$$K(x,\bar{y}) \ngeq K(\bar{x},\bar{y}) \text{ for all } x \in X$$

and $K(\bar{x},y) \ngeq K(\bar{x},\bar{y})$ for all $y \in Y$

DEFINITION 2. (Tanino,Nakayama and Sawaragi[12]): Let $f: R^n \to R^p$. A point $\bar{x} \in X$ is said to be an efficient solution of the vector maximization problem: V-max f(x) over $x \in X$, if there does not exists any $x \in X$ such that $f(x) \geq f(\bar{x})$.

DEFINITION 3. (Rodder [10])

A point $(x^0, y^0) \in X \times Y$ is called a solution of the max-min problem if

(i) y^0 is an efficient solution of $V-\min_{y\in Y} K(x^0, y)$

(ii) $K(x^0,y^0) \nleq K(x,y)$ for all $x \in X$ and $y \in Y$.

DEFINITION 4. (Rodder [10]) A point $(x^0, y^0) \in X \times Y$ is called a solution of the min-max problem if

(i) x^0 is an efficient solution of $V-\max_{x\in X} K(x, y^0)$.

(ii) $K(x^0,y^0) \ngeq K(x,y)$ for all $x \in X$ and $y \in Y$.

DEFINITION 5. (Rodder [10]) A point $(x^0, y^0) \in X \times Y$ is called a generalized saddle point if (x^0, y^0) solves both max-min and min-max problems.

LEMMA 1. (Rodder [10]) :The following statements are equivalent.

(i) (x^0, y^0) is a generalized saddle point of *K(x,y)*in $X \times Y$.

(ii) y^0 solves $V-\min_{y\in Y}(x^0,y)$ and x^0 solves $V-\max_{x\in X} K(x,y^0)$.

(iii) $K(x,y^0) \ngeq K(x^0,y^0) \forall x \in X$ and $K(x^0,y) \nleq K(x^0,y^0) \forall y \in Y$

We now state the following two multi-objective programming problems (P) and (D) and establish the main duality theorem in Section s:

(P): $V-\min(K_1(x,y)-x^T\nabla_1[\mu^T K(x,y)],\ldots,K_p(x,y)-x^T\nabla_1[\mu^T K(x,y)]),$

subject to

$$\nabla_1[\mu^T K(x,y)] \nleq 0, \qquad (1)$$

$$x \ngeq 0, y \ngeq 0, \mu \in \wedge. \qquad (2)$$

(D): $V-\min(K_1(u,v)-x^T\nabla_2[\mu^T K(u,v)],\ldots,K_p(u,v)-x^T\nabla_2[\mu^T K(u,v)]),$

$$\nabla_2[\alpha^T K(u,v)] \geq 0, \qquad (3)$$

$$u \geq 0, v \geq 0, \alpha \in \wedge. \qquad (4)$$

Here $x,u \in R^m; y,v \in R^n; \mu,\alpha \in R^p;$ and $K: R^m \times R^n \to R^p$.

3. Vector-valued infinite game and multi-objective programming

Corresponding to the multi-objective programming problems (P) and(D) as defined above, we introduce the vector-valued infinite game {VG:ST,K}.

where

(i) $S = \left\{x \in R^m : x \geq 0\right\}$ is the strategy space for player I,

(ii) $T = \left\{ y \in R^n : y \geq 0 \right\}$ is the strategy space for player II.

and

(iii) $K : S \times T \to R^p$ defined by K(*x,y*) is the payoff to player I. The payoff to player II will be taken as-K(*x,y*).

The theorems given below give necessary and sufficient conditions for a pair $(\bar{x}, \bar{y}) \in S \times T$ to be an equilibrium point of the game VG.

THEOREM 1. (Necessary conditions): Let $(\bar{x}, \bar{y})$ be an equilibrium point of the game VG. Then there exists $\bar{\mu} \in R_+^p, \bar{\mu} \neq 0$ and $\bar{\alpha} \in R_+^p, \bar{\alpha} \neq 0$ such that and $(\bar{x}, \bar{y}, \bar{\mu})$ and $(\bar{x}, \bar{y}, \bar{\alpha})$ are efficient to multi-objective programming problems (P) and (D) respectively.

PROOF. Since $(\bar{x}, \bar{y})$ is an equilibrium point of the game VG, it follows that

$$K(\bar{x}, \bar{y}) \not\leq K(x, \bar{y}) \qquad \forall x \in S \qquad (5)$$

$$K(\bar{x}, \bar{y}) \not\geq K(\bar{x}, y) \qquad \forall y \in T \qquad (6)$$

Now (5) implies that $\bar{x}$ is an efficient solution of the following:

$$(P)_{\bar{y}} V - \max K(x, \bar{y}), \text{ subject to } x \geq 0.$$

Hence there exists (Singh [11]) $\mu_0 \in R_+^p, \mu \neq 0$ such that

$$\nabla_1 [\mu_0^T K(\bar{x}, \bar{y}) \leq 0,$$

$$x^{-T} \nabla_1 [\mu_0^T K(\bar{x}, \bar{y}) = 0,$$

$$\bar{x} \geq 0.$$

Let $\bar{\mu} = (\mu_0 / \mu_0^T e)$ so that $\bar{\mu} \in \wedge$.

Since $\bar{y} \in T$, it follows that $(\bar{x}, \bar{y}, \bar{\alpha})$ is feasible for (P) with $x^{-T} \nabla_1 \mu^{-T} K(\bar{x}, \bar{y}) = 0$. Now it remains to show that $(\bar{x}, \bar{y}, \bar{\mu})$ is efficient to (P). If possible let $(\bar{x}, \bar{y}, \bar{\mu})$ be not efficient to (P); then there exists (x_0, y_0, μ) which is feasible for (P) such that

$$K_i(x_0, y_0) - x_0^T \nabla_1 [\mu^T K(x_0, y_0)] \leq K_i(\bar{x}, \bar{y}) - x^{-T} \nabla_1 [\mu^{-T} K(x, y)]$$

and

$$K_j(x_0, y_0) - x_0^T \nabla_1 [\mu^T K(x_0, y_0)] < K_j(\bar{x}, \bar{y}) - \bar{x}^T \nabla_1 [\bar{\mu}^T K(x, y)]$$

for at least j.

The above relations give $K(x_0, y_0) \leq K(\bar{x}, \bar{y})$ which contradicts the definitions of a generalized saddle point. Hence $(\bar{x}, \bar{y}, \bar{\mu})$ is efficient to (P). Similarly form (6), we get that $(\bar{x}, \bar{y}, \bar{\alpha})$ is efficient to (D).

THEOREM 2. (Sufficient conditions): Let $(\bar{x}, \bar{y}, \bar{\mu})$ and $(\bar{x}, \bar{y}, \bar{\alpha})$ be feasible for (P) and (D) respectively with

$$\bar{x}^T \nabla_1 [\mu^T K(\bar{x}, \bar{y})] = 0 = \bar{y}^T \nabla_1 [\bar{\alpha}^T K(\bar{x}, \bar{y})]$$

and $\bar{\mu} > 0, \bar{\alpha} > 0$. Also let, for each i=1,2…..,p, K_i be concave-convex. Then $(\bar{x}, \bar{y})$ is an equilibrium point of the game VG.

Proof:

We have to prove that

$$K(\bar{x}, \bar{y}) \not\leq K(x, \bar{y}) \qquad \forall x \in S.$$

$$K(\bar{x}, \bar{y}) \not\geq K(\bar{x}, y) \qquad \forall y \in T.$$

If possible, let $K(\bar{x},\bar{y}) \leq K(\hat{x},\bar{y})$ for some $\hat{x} \in S$. Therefore $\bar{\mu}^T K(\bar{x},\bar{y}) < \bar{\mu}^T K(x,\bar{y})$.

Now by the concavity of $\bar{\mu}^T K$ at x, we have

$(\hat{x}-\bar{x})^T \nabla_1[\bar{\mu}^T K(\bar{x},\bar{y})] > 0$

i.e.

$$\hat{x}^T \nabla_1[\bar{\mu}^T K(\bar{x},\bar{y})] > \bar{x}^T \nabla_1[\bar{\mu}^T K(\bar{x},\bar{y})]. \tag{7}$$

But (3) together with the hypothesis of the theorem yields

$$x^T \nabla_1[\bar{\mu}^T K(\bar{x},\bar{y})] > 0,$$

which contradicts (1). Hence $K(\bar{x},\bar{y}) \nleq K(x,\bar{y}), \forall x \in S.$ Similarly we can show that $K(\bar{x},\bar{y}) \nleq K(\bar{x},y), \forall y \in T.$

Corollary1. If $\mu \geq 0$ and each K_i is strictly concave at $\bar{x}$, then Theorem 2 holds also.

Corollary 2. If $\alpha \geq 0$ and each K_i is strictly convex at $\bar{y}$, then Theorem 2 holds also.

4. Symmetric duality

In this section, we shall prove a symmetric duality theorem for multi-objective programming problems (P) and (D). In this context, it may be remarked that the traditional weak duality theorem [13] does not holdgood for multi-objective programming problems (P) and (D), as illustrated by the following example.

Example: Let

$$K_1(x,y) = -x_1^2 - 30x_2^2 + 2y_1^2 + 50y_2^2.$$
$$K_2(x,y) = -3x_1^2 - 0.5x_2^2 + 5y_1^2 + 0.4y_2^2.$$

where $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ then

$$(x_1 = 0.2, x_2 = 0.3, y_1 = 0.0, y_2 = 0.0, \mu_1 = 0.25, \mu_2 = 0.75)$$

and $(u_1 = 0.0, u_2 = 0.0, v_1 = 1.0, v_2 = 0.0, \alpha_1 = 0.5, \alpha_2 = 0.5)$ are feasible solutions for (P) and (D) respectively. Further for these feasible solutions, the primal and dual objective values for (P) and (D) are (-1.1225, 1.4525) and (-1.0, 2.0) respectively. But -1.1225<-1.0 and 1.4525<2.0, and so the weak duality theorem between (P) and (D) does not hold good.

THEOREM 3.(Symmetric Duality): Let $(\bar{x},\bar{y},\bar{\mu})$ be an efficient solution of (P) with $\bar{\mu} > 0$. Assume that the Hessian matrix $\nabla_{11}[\bar{\mu}^T K]$ is negative definite. Let for each $i = 1,2,...,p, K_i(.,\bar{y})$ be concave at $\bar{x}$ and $K_i(\bar{x},.)$ be strictly convex at $\bar{y}$. Then there exists $\bar{\alpha} \in R_+^p, \bar{\alpha} \neq 0$ such that $(\bar{x},\bar{y},\bar{\alpha})$ is efficient to (D).

Proof:

Since $(\bar{x},\bar{y},\bar{\mu})$ is an efficient solution of (P) , it is a weak minimum. Hence there exists $\xi \in R^p, \delta \in R^m, \beta \in R^n, \gamma \in R^p, \eta \in R^p$ such that $(\bar{x},\bar{y},\bar{\mu})$ satisfies the following conditions ([3], and [4]); (For simplicity we write $\nabla_1[\bar{\mu}^T K], \nabla_{11}[\bar{\mu}^T K]$ etc. instead of $\nabla_1[\bar{\mu}^T K(\bar{x},\bar{y})], \nabla_{11}[\bar{\mu}^T K(\bar{x},\bar{y})]$ etc. respectively.

$$\nabla_1\left[\xi^T-\left(\sum_i \xi_1\right)\bar{x}^T K\right]\nabla_{11}\left[\bar{\mu}^T K\right]+\delta^T\nabla_{11}\left[\bar{\mu}^T K\right]-\left(\sum_i \xi_1\nabla_1\left[\bar{\mu}^T K\right]\right)\geq 0. \tag{8}$$

$$\bar{x}^T\nabla_1\left[\xi^T K\right]-\left(\sum_i \xi_1\right)\bar{x}^T\left[\nabla_{11}[\bar{\mu}^T K]\right]\bar{x}+\delta^T\nabla_{11}\left[\bar{\mu}^T K\right]\bar{x}-\left(\sum_i \xi_1\right)\bar{x}^T\nabla_1[\bar{\mu}^T K]=0. \tag{9}$$

$$\nabla_2\left[\xi^T K\right]-\left(\sum_i \xi_1\right)\bar{x}^T\nabla_{12}\left[\bar{\mu}^T K\right]+\delta^T\nabla_{12}\left[\bar{\mu}^T K\right]-\beta=0, \tag{10}$$

$$-\left(\sum_i \xi_1\right)\bar{x}^T\nabla_1 K_i+\delta^T\nabla_1 K_i-\gamma_i-\eta=0, \qquad i=1,2,......,p, \tag{11}$$

$$\delta^T\nabla[\mu^{-T}K]=0, \tag{12}$$

$$\beta^T\bar{y}=0, \tag{13}$$

$$\gamma^T\bar{\mu}=0, \tag{14}$$

$$(\xi,\delta,\beta,\gamma)\neq 0, \tag{15}$$

$$(\xi,\delta,\beta,\gamma)\geq 0 \tag{16}$$

Since $\bar{\mu}>0$, it follows from [14] that $\gamma=0$. Hence (11) becomes

$$\left(\delta-\sigma\bar{x}\right)^T\nabla_1 K_i-\eta=0, \qquad i=1,2,....,p \tag{17}$$

where $\sigma=\sum_{i=1}^{p}\xi_i$,(8) and (9) can be rewritten as

$$\nabla_1\left[(\xi-\sigma\bar{\mu})^T K\right]+(\delta-\sigma\bar{x})^T\nabla_{11}[\bar{\mu}^T K]\geq 0, \tag{18}$$

$$\bar{x}^T\nabla_1\left[(\xi-\sigma\bar{\mu})^T K\right]+(\delta-\sigma\bar{x})^T\nabla_{11}[\bar{\mu}^T K]\bar{x}=0. \tag{19}$$

Now from (16) ,(18) and (19), it follows that

$$(\delta-\sigma\bar{x})^T\nabla_1\left[(\xi-\sigma\bar{\mu})^T K\right]+(\delta-\sigma\bar{x})^T\nabla_{11}[\bar{\mu}^T K](\delta-\sigma\bar{x})^T\geq 0.$$

By using, (17) the above inequalities gives

$$\left(\sum_i \xi_i-\sigma\sum_i \mu_i\right)\eta+(\delta-\sigma\bar{x})^T\nabla_{11}[\bar{\mu}^T K](\delta-\sigma\bar{x})^T\geq 0, \text{ which implies that}$$

$$(\delta-\sigma\bar{x})^T\nabla_{11}[\bar{\mu}^T K](\delta-\sigma\bar{x})>0.$$

Since the Hessian matrix $\nabla_{11}[\bar{\mu}^T K]$ is negative definite, it follows that

$$\delta-\sigma\bar{x}=0$$

$$\Rightarrow \quad \delta=\sigma\bar{x} \tag{20}$$

Let $\sigma=0$. Then $\xi=0$ and $\delta=0$. Thus from (10) and (17) , we have $\beta=0$ and $\eta=0$. Hence $\xi=0,\delta=0,\gamma=0,\ \beta=0,\eta=0$ contradicts (15). Therefore, $\sigma>0$, i.e. $\xi\geq 0$. Now (10) and (20) imply

$$\nabla_2[\bar{\xi}^T K]\geq 0 \text{ where } \bar{\xi}=\xi/\sigma. \tag{21}$$

Also, (10),(13) and (20) give

$$\bar{y}^T\nabla_2[\bar{\xi}^T K]=0. \tag{22}$$

Thus from (21) and (22), it follows that $(\bar{x},\bar{y},\bar{\xi})$ is feasible for (D) with $\bar{y}^T\nabla_2[\bar{\xi}^T K]=0$.

Also, from (20) and (12), we have

$$\bar{y}^T\nabla_2[\bar{\xi}^T K]=0.$$

Now by applying Theorem 2, we have $(\bar{x}, \bar{y})$ is an equilibrium point of the game VG. Hence by Theorem 1, there exists $\bar{\alpha} \in R_+^p, \bar{\alpha} \neq 0$ such that $(\bar{x}, \bar{y}, \bar{\alpha})$ is efficient to (D). This proves Theorem 3.

REFERENCES

1. **H.W. Corley**,
"Games with vector pay-offs", Journal of Optimization theory and applications 47 (1985) 491-498.
2. **R.W.Cottle**,
"An infinite game with a convex-concave pay-off kernel",Research Report No. ORC 63-19(RN-2), Operation Research Centre, University of California, Berkley, 1963 Unpublished.
3**. B.D.Craven**,
"Lagrangian conditions and quasi-duality", Bull. Aust. Math. Soc. 16(1977) 325-339.
4. **B.D.Craven**,
"Lagrangian conditions, vector minimization and local duality", Research Report No. 37, Department of Mathematics, University of Melbourne, Australia, 1980.
5. G.B.Dantzig,E. Eisenberg and R.W. Cottle,
"Symmetric dual nonlinear program", Pacific J.Math.15(1965), 809-812.
6**. W.S.Dorn**,
"A symmetric dual theorem for quadratic programs"'Journal of Operations Research Society ofJapan 2 (1960) 93-97.
7. **B.Mond**,
"A symmetric dual theorem for nonlinear programs", Quart of Appl. Math. 23(1965) 265-269.

8. **B.Mond**,
"A symmetric duality for nonlinear programming",OPSEARCH 13 (1976) 1-10.
9. **B.Mond and T. Weir,**
"Generalized concavity and duality", in Generalized Concavity Optimization asnd Eckonomics (eds. S.Schaible and W.T.Zeimbai),(Academic Press,1981),263-279.

10. **W.Rodder**,
" A generalized saddle point theory: its application to duality theory for linear vector optimum problems", European Journal of Operation Research (1977), 55-59.
11. **C.Singh**,
"Optimality conditions in multiobjective differentiable programming", Journal of Optimization Theory and Application53(1987) 115-123.
12**. T.Tanino,Y.Sawargi and A.Nakayama**,
"Theory of Multiobjective Optimization (Academic Press.Inc.USA,1985.)

13. **T.Weirand B.Mond**,
"Symmetric and self duality in multiobjective programming", Asia-Pacific Journal of Operational Research 5(2) (1988) 124-133.